# Bayesian Estimation of Asymmetric Jump-Diffusion Processes *

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**ABSTRACT**

The hypothesis that asset returns are log-normally distributed has been widely rejected. The extant literature has shown that empirical asset returns are highly skewed and leptokurtic (fat tails). The Affine Jump-Diffusion (AJD) model improves upon the log-normal specification by adding a jump component to the return process. The two-sided jump-diffusion (TSJD) model further improves upon the AJD specification by allowing for the tail behavior of the return distribution to be *asymmetrical*. The Pareto-Beta (Ramezani and Zeng, 1998) and the Double Exponential (Kou, 2002) models present two alternative TSJD specifications. Under the Pareto-Beta specification, two Poisson processes govern the arrival rate of good and bad news, leading to Pareto distributed up-jumps or Beta distributed down-jumps in prices. Under the Double Exponential specification, a single Poisson process generates jumps in returns but the up and down magnitudes are generated by two exponential distributions. Both specifications results in highly asymmetric jump diffusion processes with desirable empirical and theoretical features. Accordingly, these models have been widely adopted in the portfolio choice, option pricing, and other branches of the literature. The primary objective of this paper is to contribute to the econometric methods for estimating the parameters of the TSJD models. Relying on the Bayesian approach, we develop a Markov Chain Monte Carlo (MCMC) estimation technique that is appropriate to these specifications. We then provide an empirical assessment of these model using daily returns for the S&P-500 and the NASDAQ indexes, as well as individual stocks. We complete our analysis by providing a comparison of the estimated parameters under the MCMC and the MLE methodologies.

**Keywords:** Asset Price Processes, Affine Jump-Diffusion, Double Exponential Jump-Diffusion, Markov Chain Monte Carlo, Bayesian Econometrics

**JEL Classification: C32, C52, G12, G13**

# 1 Introduction

Almost every aspect of modern finance theory, from valuations and portfolio choice to option pricing and corporate finance, as well as the ever expanding field of mathematical finance, critically depends upon the form of the probability distribution describing the dynamics of security prices or other underlying value drivers. Although the Geometric Brownian Motion (GBM) had served as a convenient paradigm for some time, as the empirical evidence against GBM accumulated, the Affine Jump-Diffusion (AJD) representation, pioneered by Merton (1976), gained wide acceptance primarily because it was shown to be consistent with the empirical features of asset returns (higher mode and excess kurtosis and skewness). Moreover, a large body of empirical evidence indicates that option pricing formulas based on AJD representation exhibit better pricing accuracy. That is, the AJD specifications better explains the volatility "smile" and "skew" across different moneyness and maturity.

Given this paradigm shift, the AJD specification now serves as the starting point in most dynamic portfolio choice and asset valuation models (Duffie et al, 2000, 2003).[1] The popularity of the AJD framework is due to its modeling flexibility that captures important features of financial risk processes, its technical tractability in deriving standard and extended transforms for option and bond pricing, and econometric estimation. Moreover, empirical evidence indicates that these models offer superior fit to the data.

In its simplest form, an AJD process consists of three basic building blocks; a drift, a Brownian motion representing normal price variations, and a jump process that captures extreme movements in prices. While the simple AJD specification admits a leptokurtic and asymmetric return distribution, it has proved to be inadequate in fully matching the sample moments of asset returns. This shortcoming has resulted in further generalizations of the AJD class of representations. This is achieved by assuming different theoretical structures for the drift (e.g., mean reversion), the diffusion (e.g., stochastic volatility), and the jump component (e.g., different distributions for the jump magnitude, time varying jump intensity, and correlated jumps).[2]

In its most popular form, the AJD price process has a single jump component that captures the impact of news on security prices (Merton, 1976). News that lead to positive jumps in prices –"good news"– and news that lead to

[1] Surveying this body of literature represents a daunting task that is outside the scope of this paper. It suffices to note that AJD specification has been used to model the risk underlying interest bearing instruments, foreign exchange, real assets (e. g. commodities), and stocks and indexes. Recently, the AJD framework has also become integral to the expanding literature concerned with the valuation of insurance products (mortality risk), and credit risk modeling. Cont and Tankov (2003) discuss the applications of AJD specification and the papers in Aït-Sahalia and Hansen (2004) provide a complete survey of the most important developments, focusing on the econometric issues. Carr and Wu (2004) provide an overview of the application of these models for valuation of options and other contingent claims.

[2] Cont and Tankov (2003) provide a comprehensive survey of this literature, focusing on models of the jump component.

negative jumps in prices –"bad news"– are not distinguished by their intensity or distributional characteristics. This potential limitation of the simple jump-diffusion framework has led to two alternative specifications. Under Pareto-Beta Jump-Diffusion (PBJD), proposed by Ramezani and Zeng (1998), good and bad news are generated by two independent Poisson processes and jump magnitudes are drawn from the Pareto and Beta distributions. Alternatively, under the Double Exponential Jump Diffusion (DEJD), proposed by Kou (2002), a single Poisson process with fixed intensity generates news, but the jump magnitudes representing abnormal up- and down-price movements are drawn from two independent exponential distributions. As Ramezani and Zeng (2007) have shown, the two models are closely related in that the parameters of one model may be retrieved from the other. Given the close kinship between these models, henceforth we will refer to them as the Two-Sided Jump-Diffusion (TSJD) specifications.

The TSJD representation has gained popularity primarily because of its distributional flexibility. Furthermore, as Kou (2007) and others have shown, the TSJD specifications lead to nearly analytical option pricing formulas for certain exotic and path dependent options. This is a significant advantage as most of the existing methods for pricing options under the jump-diffusion processes are confined to plain vanilla European options. Because of these and other advantageous features (Kou, 2007, p. 86), the applications of the TSJD framework has been expanding in the literature. Kou (2007) and Ramezani and Zeng (2007) provide a survey of important applications of TSJD framework. More recent extensions of the TSJD representations to other areas of economics and finance include Bertrand and Prigent (2011), Bo et al (2012), Cara et al (2010), Dao and Jeanblanc (2006), Deng et al (2012), Dotsis et al (2006), Moazeni et al (2011), and Zhang et al (2012).

Despite the growing interest in TSJD specification, estimation and empirical assessment of this model has received sparse attention to date. In practice, most studies have arbitrarily assumed "reasonable" parameter values under each specification and proceeded to carry out their intended modeling analysis. A notable exception is Ramezani and Zeng (2007), who utilize maximum likelihood estimation (MLE) to provide parameter estimates for the TSJD specification using daily data for individual stocks, the S&P-500, and the NASDAQ composites. The empirical tests performed by these authors suggests that the TSJD specification provides a superior fits to returns data, relative to Merton's (1976) single jump, as well as the classical GBM specifications.[3]

It is well known that maximum likelihood estimation of AJD type models, including TSJD, offers a number of theoretical advantages (Sorensen, 1991). In particular, the MLE estimates are asymptotically consistent, normal, and efficient. Moreover, their standard errors can be obtained from the estimator's asymptotic variance-covariance

[3]For a survey of estimation procedures for AJD class of representations see Zhao (2008), Singleton (2006), and the papers in Aït-Sahalia and Hansen (2004).

matrix.[4] Lastly, Aït-Sahalia (2004) has shown that MLE offers advantages in "disentangling" the jumps from the diffusion component of returns.

While the asymptotic properties of MLE are highly desirable, a number of challenges limit its application to the AJD class of models. First, MLE implementation requires a complete specification of the transition density, which for nonlinear models (e.g., stochastic volatility with jumps) may be difficult to obtain. Second, for the AJD specification, the unconditional density of returns will be a mixture density and unless the parameter space is properly restricted, singularities may arise and the log-likelihood function may be unbounded (Honoré, 1998). Third, the likelihood function involves infinite summations and improper integrals, resulting in high computational costs and difficulty in identifying the global optimum. Furthermore, the MLE estimates and their standard errors are highly sensitive to the choice of the numerical optimization procedure. Lastly, the desirable properties of MLE are predicated on having a sufficiently "large sample" size and such properties are unlikely to remain valid with limited data. Given the aforementioned challenges, several alternative methods for the estimation of the AJD processes have been proposed (Aït-Sahalia and Hansen, 2004; Zhao, 2008). These alternatives possess advantages and shortcomings of their own, which is the subject of a large body of theoretical and empirical literature.

The primary objective of the present paper is to contribute to the extant literature that is concerned with the estimation of the TSJD models. Specifically, we extend the existing Bayesian estimation procedures, that have been developed for the AJD class, to the TSJD specifications. In particular, we extend the Markov Chain Monte Carlo (MCMC) estimation technique that were developed for the AJD specification (Johannes and Polson, 2009) to the TSJD models. We then provide a comparison of the estimated parameters for the TSJD specifications under MCMC and MLE. To our knowledge, this is the first such comparison. We use daily returns for the S&P 500 (1962-2011), the NASDAQ (1973-2011), and individual stocks. We find that these estimation procedures result in similar parameter estimates. However, the standard errors from the Bayesian procedure are significantly smaller. Moreover, the MCMC implementation is computationally efficient and much faster. In Section (2), we briefly present the details of the TSJD specifications. In Section (3), we discuss the required steps in MCMC estimation of these models. In Section (4),we test the MCMC estimation method using simulated data. Section (5) presents the empirical results and provides a comparison of the MCMC and the MLE estimation techniques. The paper concludes with suggestions for further enhancements to this line of research.

[4] In practice, given that the "true" parameters are unknown, a consistent estimator of the variance-covariance matrix is used. One such estimator is obtained by the outer-product method, which is based on the first derivative of the likelihood function (Hamilton, 1994, p. 143).

## 2 Two-Sided Jump Diffusion Models

In this section, we present the Perato-Beta Jump Diffusion (PBJD) and the Double Exponential Jump Diffusion (DEJD) models for the return processes. The PBJD posits that return process contains two jump components, representing good and bad news arrival. Each type of news is generated by independent Poisson processes that lead to up- or down-jumps in prices, whose magnitudes are drawn from the Pareto and Beta distributions, respectively. This formulation is consistent with Milgrom (1981), who formalized the notion of "good" and "bad" news and showed that the arrival of good (bad) news about a firm's prospects will always lead to a rise (fall) in its share price. Ramezani and Zeng (1998) present other plausible economic justifications for invoking a distinction between good and bad news. They note that at the individual stock level, discontinuous up- and down-price movements may be a consequence of significant changes in the operations and the financial structure of the firm, its competitive environment, unexpected changes in its organizational form, and sudden shifts in its corporate plans. Similarly, at the stock index level, macroeconomic policy decisions, such as a cut (increase) in interest rates by the Federal Reserve, serves as the unexpected good (bad) news that leads to an up- (down-) jump in index value.[5]

The separation of good from bad news implies that the range of values for the random percentage change in prices must be constrained. Because stocks represent limited liability, the percentage change in price due to bad news must be bounded from below by minus one hundred percent. Similarly, the percentage change in price due to arrival of good news must be positive. To capture these restrictions, the jump magnitudes for good and bad news are drawn from the Pareto and Beta distributions, respectively.

Let $S(t)$ denote the price of stock at time $t$, the PBJD process can be represented by:

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dZ(t) + \sum_{j=u,d} (V^j_{N_j(\lambda_j t)} - 1) dN_j(\lambda_j t)$$

where $\mu$ and $\sigma$ are the drift and volatility terms, $Z(t)$ is a standard Wiener process, $V^j$ is the jump magnitude, and $N_j(\lambda_j)$ are independent Poisson processes with intensity parameters $\lambda_j$ ($j = u, d$ represent up- and down-jumps respectively). It is assumed that the up-jump magnitudes $V^u$ are distributed Pareto($\eta_u$) and the down-jump magnitudes $V^d$ are distributed Beta($\eta_d$,1). Letting $Y = ln(V)$, Ramezani and Zeng (2007) show that the distribution of $Y$ is:

$$f_Y(y) = (\frac{\lambda_u}{\lambda_u + \lambda_d})\eta_u e^{-\eta_u y} I\,(y \geq 0) + (\frac{\lambda_d}{\lambda_u + \lambda_d})\eta_d e^{\eta_d y} I\,(y < 0)$$

[5] Maheu and McCurdy (2004) show that expansionary and contractionary economic periods are accompanied with unequal frequency of good and bad news arrivals. In our results section, we provide direct evidence in support of this conjecture using recent data that span a dramatic bull and bear market periods (2007-2010).

Let $\underline{r} = (r_1, \cdots, r_M)$ denote a realization of the one period rate of return ($r_i = \ln(\frac{S(i)}{S(i-1)})$), at equally-spaced times $i = 1, 2, \ldots, M$. Then under the PBJD model, one period return is IID and the unconditional density of returns, $f(r|\theta_{PBJD})$, is:

$$f(r\,|\theta_{PBJD}) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P(m|\lambda_d)P(n|\lambda_u)f_{n,m}(r|\mu_{n,m}, \sigma^2_{n,m}) \tag{1}$$

where $\theta_{PBJD} = (\mu, \sigma, \lambda_u, \lambda_d, \eta_u, \eta_d)$ and $P(j|\lambda) = \frac{e^{-\lambda}\lambda^j}{j!}$. The density $f(r\,|\theta_{PBJD})$ is the Poisson weighted mixture of the conditional densities, $f_{n,m}(r|\cdot,\cdot)$, given $n$ up-jumps and $m$ down-jumps, with the conditional mean and the conditional variance defined as:

$$\mu_{n,m} = (\mu - \frac{1}{2}\sigma^2 + \frac{n}{\eta_u} - \frac{m}{\eta_d})$$
$$\sigma^2_{n,m} = (\sigma^2 + 2(\frac{n}{\eta_u^2} + \frac{m}{\eta_d^2}))$$

The conditional density, $f_{n,m}(r|\cdot,\cdot)$, is derived in Ramezani and Zeng (2007), who show that the log-likelihood function for the PBJD models involves double infinite summations and double improper integrals.

Using the proceeding notation, the DEJD process can be represented by the following

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dZ(t) + \sum_{i=1}^{N(t)}(V_i - 1)dN(\lambda t)$$

where $\mu$, $\sigma$, and $Z(t)$ are defined as above, and $N(\lambda)$ is an independent Poisson processes with intensity parameter $\lambda$ and $V_i$ is a sequence of IID random variables. Rather than specifying a distributions for $V$, Kou (2002) assumes that $Y = ln(V)$ has an IID mixture distribution of Exponential ($\eta_u$) with mixing probability $p$ and Exponential ($\eta_d$) with probability $q = 1 - p$:

$$f_Y(y) = p\eta_u e^{-\eta_u y} I\,(y \geq 0) + q\eta_d e^{\eta_d y} I\,(y < 0)$$

Under the DEJD formulation, the unconditional density of one period return is:

$$f(r|\theta_{DEJD}) = \sum_{k=0}^{\infty} P(k|\lambda)f_k(r|\mu_k, \sigma^2_k) \tag{2}$$

where $\theta_{DEJD} = (\mu, \sigma, \lambda, p, \eta_u, \eta_d)$ and the density $f(r|\theta_{DEJD})$ is the Poisson weighted mixture of the conditional densities, $f_k(r|\cdot,\cdot)$, given $k$ jumps, with the conditional mean and the conditional variance defined as:

$$\mu_k = \mu - \frac{1}{2}\sigma^2 + k(\frac{p}{\eta_u} - \frac{q}{\eta_d})$$
$$\sigma^2_k = \sigma^2 + k(pq(\frac{1}{\eta_u} + \frac{1}{\eta_d})^2 + (\frac{p}{\eta_u^2} + \frac{q}{\eta_d^2}))$$

Both TSJD specifications are Lévy processes with stationary and independent increments that is continuous in probability. Considering the two alternatives, Ramezani and Zeng (1998) have shown that the parameters one model may be retrieved from the other by imposing the following mathematical restrictions: $\lambda = \lambda_u + \lambda_d$, $p = \frac{\lambda_u}{\lambda}$. The natural interpretations for these restrictions is that the total jump rate for the Poisson process governing news arrival is simply the sum of the *independent* up- and down-jump Poisson news arrival rates. Moreover, the probability of a draw from the upper tail of the double exponential distribution is determined by the relative arrival rate of the up-jumps to total arrival rate. Note that while both TSJD specifications have the same number of parameters, the estimated parameters may not conform with the mathematical restrictions needed to recover one model's parameters from the other, unless the "true" data generating mechanism is consistent with the assumed distributions and the noise structure is identical. Still, both models are capable of generating a higher peak, positive or negative skewness, and positive kurtosis. Therefore, both models are likely to better match the empirical moments of returns.

Parameter estimation by MLE is a challenging task for both models, because of the well known complications associated with the likelihood function of mixture distributions. The MLE optimization requires evaluation of infinite summations and integration of improper integrals. In practice, both the summations and the integrals are truncated using an appropriate convergence criteria. Moreover, the likelihood function of the mixture distribution may explode unless the parameter space is "appropriately" restricted. It is important to note that because the Bayesian method utilizes the likelihood function, these issues also carry over to the MCMC estimation. As the extensive literature in Bayesian statistics demonstrates, that the MCMC algorithm provides an attractive numerical way to deal with the intractable integration problems that hinder the MLE approach. Moreover, MCMC provides simple diagnostics to avoid the problem of an unbounded likelihood function and is computationally efficient. The advantages offered by MCMC provides an important impetus for our analysis.

# 3 Parameter Estimation via MCMC

Bayesian estimation methods have become more attractive and broadly used in finance and economic research (Johannes and Polson, 2009). Bayesian estimation methods that are well suited to the AJD processes have been widely studied by Li et al (2008), Goncalves and Roberts (2010), Jacquier et al (2007), and Johannes and Polson (2009). While MLE provides parameter estimates with desirable properties, for complicated models like TSJD, it may not yield reliable standard errors. Modern computational capabilities allow Bayesian methods to obtain results as fast as MLE, but offer significant improvement in obtaining standard errors.

MCMC is a conditional simulation algorithm that generates random sample from a target distribution, $f(\theta_{PBJD}, X|\underline{r})$, where $X = \{N_u(t), N_d(t), Y_i^u, Y_i^d\}$ are the latent state variables. The main idea behind MCMC is that the target joint distribution can be characterized by the complete set of conditional distributions. That is, knowledge of $f(\theta_{PBJD}|X, \underline{r})$ and $f(X|\theta_{PBJD}, \underline{r})$ completely characterizes the target joint distribution, $f(\theta_{PBJD}, X|\underline{r})$. The steps in the algorithm are as follows: given the initial parameter values $\theta_{PBJD}^{(0)}$, draw $X^{(1)} \sim f(X|\theta_{PBJD}^{(0)}, \underline{r})$ and then $\theta_{PBJD}^{(1)} \sim f(\theta_{PBJD}|X^{(1)}, \underline{r})$, continuing this iteration $j$ times to generate a sequence of random variables $\{X^{(j)}, \theta_{PBJD}^{(j)}\}$. This sequence is a Markov Chain with the attractive property that the equilibrium distribution of the chain converges to target distribution, $f(\theta_{PBJD}, X|\underline{r})$. In our procedure, we use the *Gibbs sampler* and *Metropolis-Hasting (MH)* algorithm to sample from $f(\theta_{PBJD}|X, \underline{r})$, because we cannot sample from this distribution directly. Both sampling algorithms provide random samples from the target joint distribution that can be used for parameter and state variable estimation using the Monte Carlo method. The details of our algorithm for the estimating the parameters of the TSJD models are presented in the table below and derived in the appendix. Li et al (2008), Johannes and Polson (2009) and Jacquier et al (2007) contain comprehensive exposition of these concepts and examples of the applications of MCMC estimation to problems in economics and finance.

The MCMC estimation requires that prior distributional beliefs for $\theta_{PBJD}$ be specified. Accordingly, a criticism of Bayesian methods is the choice and justification of the assumed prior distributions. In this study we assume vague (uninformed) priors as noted in Bishop (2006) and Hastie et al (2009). That is, we only require the parameters to exist in a sample space consistent with their support. This may be theoretically true of MLE, however, unstable gradients could lead to negative parameter values unless MLE parameters are obtained via constrained optimization.[6] Vague priors relieve us of the burden of justifying prior information choices and greatly reduces the analytical and computational complexity of the estimation problem. Assume the elements of $\theta_{PBJD}$ are mutually independent, so that the posterior distribution of $\theta_{PBJD}$ given $\underline{r}$ is:

$$\begin{aligned} f(\theta_{PBJD}|\underline{r}) &\propto L\left(\theta_{PBJD}|\underline{r}\right) f\left(\theta_{PBJD}\right) \\ &\propto L\left(\theta_{PBJD}|\underline{r}\right) f\left(\mu, \sigma^2, \lambda_u, \lambda_d, \eta_u, \eta_d\right) \\ &\propto L\left(\theta_{PBJD}|\underline{r}\right) f\left(\mu\right) f\left(\sigma^2\right) f\left(\lambda_u\right) f\left(\lambda_d\right) f\left(\eta_u\right) f\left(\eta_d\right) \end{aligned}$$

where $L\left(\theta_{PBJD}|\underline{r}\right) = \prod_{i=1}^{M} f(r_i \; |\theta_{PBJD})$ is the density of $r_i$. Despite the vague choice of assumed priors, it is difficult to analytically establish the posterior distribution, $f(\theta_{PBJD}|\underline{r})$, because of the complexities associate with $L\left(\theta_{PBJD}|\underline{r}\right)$. It should be clear that a suitable proposal distribution does not exist to sample from $f(\theta_{PBJD}|\underline{r})$ in

[6] Implementing constrained MLE is rarely done in practice, as it further complicates the estimation method.

a MH framework. The MH algorithm would be computationally unattractive and have low acceptance rates (Hastie et al, 2009).

Parameter estimation by MLE is a challenging task for TSJD models, because the likelihood function contains infinite summations and improper integrals that do not have a closed-form solutions. In practice, the summations are truncated when they do no significantly contribute to the likelihood function and the integrals are evaluated numerically. During the MLE process, the likelihood function of the mixture distribution will be utilized in some capacity, in either gradient calculations (by way of numerical first and second derivatives in a multivariate Newton's method framework) or to assess convergence (in a Expectation-Maximization (EM) framework). Unless the parameter space is constrained, numerical instability and singularity issues could adversely effect the terminal likelihood function value.

It should be noted that some of the difficulties associated with MLE will carry over to MCMC estimation. While our MCMC procedures relies on numerically evaluating the improper integrals, we do not need to evaluate the likelihood function. That is, the likelihood function is only used to develop a posterior distribution for sampling purpose only and it is not utilized during estimation. Rather than using the likelihood function to assess convergence, the MCMC procedures relies on simple diagnostics to determine if convergence has been achieved. Moreover, the posterior distribution is consistent with the support of the parameter space, so the process is naturally constrained. Finally, whereas MLE procedures only guarantee a non-decreasing likelihood function in this type of application (i.e. non-global convergence in the framework of EM), Bayesian methods ensure global convergence assuming the properties of Markov Chain are adequately satisfied.

Using the Bernoulli approximation originally proposed by Ball and Torous (1983), it is possible to significantly simplify both the MLE and MCMC estimation by approximating the likelihood function. [7] The Bernoulli approximation is widely used in the extant AJD literature. This approximation assumes that during the unit time under consideration (e.g., a day), either no-jump occurs or *a single jump* occurs . Consequently, in the resulting approximate Bernoulli mixture distribution $\lambda \in [0, 1]$ and acts as the probability of a single jump (i.e., a mixture model component probability), rather than the Poisson parameter that measures the news arrival rate, $\lambda \in R^+$. Our method allows the estimates of $\lambda_u$ and $\lambda_d$ to can exceed one, as they should.

More specifically, to obtain our approximate likelihood function, we assume three possible jump states: either no jumps occur, or *at least* one up-jump and no down-jumps occur, or *at least* one down-jump and no up-jumps occur (three mutually exclusive states). Let $N_u$ be the number of up-jumps with $N_u \sim Poisson\,(\lambda_u)$ and $N_d$ be the number of down-jumps with $N_d \sim Poisson\,(\lambda_d)$. Assume $N_u$ and $N_d$ are independent. The probabilities of the three

[7] See Kou (2002, footnote 7, page 1091)

mutually exclusive states are:

$$P\left(N_u \geq 1, N_d = 0\right) = \left(1 - e^{-\lambda_u}\right) e^{-\lambda_d} \equiv \lambda_u'$$
$$P\left(N_u = 0, N_d \geq 1\right) = e^{-\lambda_u}\left(1 - e^{-\lambda_d}\right) \equiv \lambda_d'$$
$$P\left(N_u = 0, N_d = 0\right) = e^{-\lambda_u} e^{-\lambda_d} \equiv \left(1 - \lambda_u' - \lambda_d'\right)$$

Using these probabilities, the approximate density function for one period returns under the PBJD has the following form:

$$f\left(r_i | \mu, \sigma^2, \lambda_u', \lambda_d', \eta_u, \eta_d\right) = (1 - \lambda_u' - \lambda_d') f_{0,0}\left(r_i\right) + \lambda_u' f_{1,0}\left(r_i\right) + \lambda_d' f_{0,1}(r_i) \tag{3}$$

where

$$f_{0,0}\left(r_i\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(r_i - \mu + \frac{\sigma^2}{2}\right)^2}$$

$$f_{1,0}\left(r_i\right) = \frac{\eta_u}{\sqrt{2\pi\sigma^2}} \int_0^\infty e^{-\eta_u y_i - \frac{1}{2\sigma^2}\left(r_i - y_i - \mu + \frac{\sigma^2}{2}\right)^2} dy_i$$

$$f_{0,1}\left(r_i\right) = \frac{\eta_d}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^0 e^{\eta_d y_i - \frac{1}{2\sigma^2}\left(r_i - y_i - \mu + \frac{\sigma^2}{2}\right)^2} dy_i$$

where $f_{n,m}\left(\cdot\right)$ is the density conditional on $n$ up- and $m$ down-jumps (see appendix in Ramezani and Zeng (2007)). We treat $f_{0,m} \approx f_{0,1}$ and $f_{n,0} \approx f_{1,0}$ for all $n, m \in N$. This approximate distribution, which is a trinomial mixture model, preserves the infinite summation of the jump events and allows for more than one jump per unit of time, while simplifying the probability distributions for the jump sizes. This approximation also reduces the complexity of the estimation problem and results in fast implementation of the proposed MCMC algorithm.

It may appear that our truncated approximate model does not include the $f_{n,m}\left(\cdot\right)$ component of the PBJD mixture (equation 1). That is, it appears that we do not explicitly allow for $n$ up-jumps and $m$ down-jumps in the same unit of time. Such instances are captured within our framework, as we consider the cumulative effect of $n$ up-jumps and $m$ down-jumps which may "offset" one another. For example, the $f_{0,0}$ component would account for "a wash" when the cumulative size of the down-jumps is roughly the same as the cumulative size of the up-jumps. This line of reasoning extends to the $f_{0,1}$ and $f_{1,0}$ components. Finally, this trinomial mixture model allows us to develop a straight-forward MCMC method that utilizes a *Gibbs sampler* which only requires a single MH step. The proposed framework quickly delivers estimates of $\theta'_{PBJD} = \left(\mu, \sigma^2, \lambda_u', \lambda_d', \eta_u, \eta_d, \right)$. Path averaging of the inverse of the above state probabilities provides estimates of $\lambda_u$ and $\lambda_d$, that can exceed unity. Furthermore, the estimates of $\lambda_u$ and $\lambda_d$ can be used to path average the functions $\lambda = \lambda_u + \lambda_d$ and $p = \frac{\lambda_u}{\lambda}$ to provide direct estimates of $\lambda$ and $p$ for the DEJD specification.

As noted above, the DEJD specification is an alternative parameterization of the PBJD model. Unlike the PBJD, the DEJD only has one news arrival rate ($\lambda$). We assume that no jumps or *at least one* jump can occurs in each time period. The probabilities of these two mutually exclusive events are $P(N = 0) = e^{-\lambda} \equiv \lambda'$ and $P(N \geq 1) = \left(1 - e^{-\lambda}\right) \equiv 1 - \lambda'$, respectively. This transformation insures that $\lambda$ can be larger than one. Given this assumption, the approximate distribution of under DEJD is:

$$f\left(r_i|\mu, \sigma^2, \lambda', p, \eta_u, \eta_d\right) = \lambda' f_0(r_i) + (1 - \lambda') f_1(r_i)$$

where $f_0 = f_{0,0}$ as referenced above and

$$\begin{aligned} f_1(r_i) &= p\eta_u \int_0^{\infty} e^{-\eta_u y_i} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(r_i - y_i - \mu + \frac{\sigma^2}{2}\right)^2} dy_i \\ &\quad + (1-p)\eta_d \int_{-\infty}^{0} e^{\eta_d y_i} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(r_i - y_i - \mu + \frac{\sigma^2}{2}\right)^2} dy_i \end{aligned}$$

Again, we treat $f_1 \approx f_n$ for all $n \in N$. We use a similar MCMC approach to estimate $\theta'_{DEJD} = (\mu, \sigma^2, \lambda^{'}, p, \eta_u, \eta_d)$. Path averaging of the inverse of the above state probability function will deliver an estimate of $\lambda$, which can be larger than one. As above, path averaging of $\lambda$ and $p$ can be used to provide estimates of $\lambda_u$ and $\lambda_d$ for the PBJD specification. The table below provides a summary of our MCMC algorithm for each model. Background details of our MCMC procedure is developed in the appendix.

| MCMC Procedure for PBJD and DEJD | | |
|---|---|---|
| Step | PBJD | DEJD |
| 1. Pick starting values | $\lambda'_u, \lambda'_d, \eta_u, \eta_d, \mu, \sigma^2$ | $\lambda', p, \eta_u, \eta_d, \mu, \sigma^2$ |
| 2. Sample: | $I_i, Y_i^u, Y_i^d$ for $i = 1, \cdots, M$ | $I_i, Y_i^u, Y_i^d$ for $i = 1, \cdots, M$ |
| 3. Sample: | $(\lambda'_u, \lambda'_d) \sim Dirichlet$ | $\lambda' \sim Beta$ and $p \sim Beta$ |
| 4. Sample: | $\eta_u \sim Gamma, \eta_d \sim Gamma$ | $\eta_u \sim Gamma, \eta_d \sim Gamma$ |
| 5. Sample: | $\mu \sim Normal$ | $\mu \sim Normal$ |
| 6. Sample: | $\sigma^2$ with Metropolis-Hastings | $\sigma^2$ with Metropolis-Hastings |
| 7. Final Step: | Repeat steps 2-6 until convergence | Repeat steps 2-6 until convergence |

# 4 Simulation Study

Before applying our proposed MCMC estimation with actual data, we performed Monte Carlo simulations to check the efficacy of our algorithm to estimate the parameters of these models and to obtain reliable standard errors. Undertaking the simulation study is an important step for the AJD class of models. As Johannes and Polson (2009) note, undertaking a simulation study is important because time discretization of AJD process can potentially bias the parameter estimates. Moreover, since methods for estimating jump diffusion models are not well developed, simulation enables the researcher to verify that the parameters can be reliably estimated for the given sample size.

Our simulation study uses simulated data sets of 2,000 data points. Each sample path is generated using a range of "true" parameter values under the PBJD and DEJD specifications. The number of observations and the assumed parameter values are suggestive of actual daily log-returns. As Table 1 shows, we experimented with a variety of parameter values, in particular we used jump parameters that would suggest frequent, as well as highly infrequent discontinuous movement in prices (low and high jump intensity cases). The simulated data, in conjunction with our MCMC algorithm, is used to estimate both models. We use a suitable burn-in period where we discard 25% of the generated samples. Tables 1 summarizes the results of our simulation study. The results indicate that our procedure produces accurate parameter estimates. It appears that some parameters, particularly the mean jump sizes, are estimated less precisely than others. Overall, most estimated parameters are close to their true values and have small standard errors.

As Aït-Sahalia (2004) has shown, the ability to disentangle jumps from the diffusion component of returns is critical to identifying the volatility parameter ($\sigma$) of the AJD process. Aït-Sahalia (2004) further shows that MLE is not hindered by this entanglement problem, even when jumps are frequent and small in magnitude. This is clearly not the case with the MCMC estimation and to our knowledge there exists no theoretical model that proves otherwise. Moreover, most applications of the MCMC estimation to the AJD processes appear to ignore the entanglement problem. This oversight may be due to the fact that Bayesian methods for detecting jumps are new (see Lee and Hannig (2010) and Lee (2012)) and somewhat difficult to implement. In the absence of a widely accepted Bayesian jump detection test, researchers have relied upon the outcome of their Monte Carlo simulation to justify the validity of their estimated parameters (i.e., high fraction of the simulated jumps correctly identified by the MCMC algorithm). Our study is no different in this regard. Integrating jump detection techniques into our MCMC approach is outside the present scope of this study, but it presents a promising and challenging future extension of this work.

# 5 Data and Results

The empirical analysis presented in this section has a number of objectives. First, utilizing the proposed MCMC approach, we will present the estimated parameters for the TSJD models, using time series data for individual stocks and two important indexes, spanning the period 1962-2010 and selected sub-periods. Second, we will provide comparisons with previous studies by using the same historical data (same span and frequency). Finally, we provide a direct comparison of our proposed MCMC procedure versus the MLE approach.[8]

We use daily (value weighted) returns for the S&P 500 and the NASDAQ composite indexes. The S&P 500 daily return series (dividend adjusted) spans the period 7/1962 through 12/2010 ($N = 10446$). The NASDAQ series spans the period 1/1973 through 12/2010 ($N = 7828$).[9] Finally, the data on individual stocks spans the period 1/1999 through 12/2003 ($N = 1256$).

To permit comparison with other studies, particularly the MLE analysis presented by Ramezani and Zeng (2007), we focus on daily returns for the S&P 500 and the NASDAQ composite indexes. We also use daily returns for 5 individual stocks, with large kurtosis (range of 3 to 10). The selected firms, which trade on NASDAQ and NYSE, are followed by a large number of analysts, and are highly liquid. These characteristics are important given the event driven nature of the TSJD models. We use the proposed MCMC methodology to estimate the parameter of PBJD and DEJD for each series.

Table 2 presents the sample statistics for the indexes and the individual stock returns. The large range of return values, particularly for the indexes, reflect significant booms and crashes that occurred during the sample period. All returns are highly skewed and have large kurtosis.

Table 3 reports both MLE and MCMC parameter estimates for the two models for the S&P 500 and the NASDAQ composite indexes.[10] Focusing on the parameter estimates, we find that estimates of $\mu$ and $\sigma$ are very similar across models and estimation techniques. This similarity is also true for the frequency of news arrivals, $\lambda_u$ and $\lambda_d$ ($\lambda_u^{-1}$ provides an estimate of the inter-arrival time). Turning to the mean up- and down-jump amplitudes, $\eta_u^{-1}$ and $\eta_d^{-1}$, we find that these parameter estimates are invariant to model specification under MCMC. However, under MLE, $\eta_u$ and $\eta_d$ estimates are significantly larger, implying smaller mean jump amplitudes. Finally, the standard errors of the estimate are significantly lower under MCMC across the board.

[8]Matlab (including the STAT package) is used for all computations on a Dell Precision T7400 (Dual Quad-Core Intel Xeon Processors, 3.40GHz/processor, 1600MHz FSB).

[9]No dividend adjusted series are available for NASDAQ index since few firms on this exchange pay dividends.

[10]We note that the MLE estimates are taken from the Ramezani and Zeng (2007) study.

Figure 1 presents the estimated PBJD distribution using the MCMC parameter estimates in Table 3. The top panel show the return distribution. The bottom panel decomposes the return distribution into the GBM, up-jump, and down-jump components. As the bottom panel shows, the asymmetry and leptokurtosis of the returns is captured by the estimated PBJD model. Note that the left-tail reflects the 1987 market crash and other significant drops.

Table 4 presents similar comparison of MLE versus MCMC using individual stock data. Again, we find that estimates of $\mu$ and $\sigma$ are very similar across models and estimation techniques. However, we find that estimates of the jump components, $\lambda_u$, $\lambda_d$, $\eta_u$ and $\eta_d$, vary significantly across models and estimation techniques. This is to be expected as total volatility for stocks contains a significant idiosyncratic component and stock returns are highly skewed and leptokurtic.

The foregoing comparisons of MCMC and the MLE estimates used historical data without regards to prevailing market conditions (Bull or Bear phases of the markets). As Ramezani and Zeng (2007) observed, the relative magnitude of the jump parameters, $\lambda_u$, $\lambda_d$, $\eta_u$ and $\eta_d$, can lead to positive (Bull) and negative (Bear) drift for the return processes. To explore this conjuncture, we focus on highly volatile recent data, spanning the period May 2007 through December 2010. We use our MCMC procedure to estimate the parameters of PBJD and DEJD for three periods: The Bear Market period (05/2007 through 03/2009), the Bull Market period (04/2009 through 12/2010), and the combined cycle (05/2007 through 12/2010).

Table 5 reports the results for the recent period and the Bull and the Bear sub-periods for the S&P 500 and the NASDAQ indexes. First note that the drift ($\mu$) and the volatility of the Brownian motion ($\sigma$) are comparable across all three epochs. We find the down-jump arrival rate, $\lambda_d$, is significantly larger than the up-jump arrival rate, $\lambda_u$, during the Bear period. During this same period, the mean jump amplitudes, $\eta_u^{-1}$ and $\eta_d^{-1}$, are similar but notably larger than the combined and the Bull period. These results confirm the conjecture of Ramezani and Zeng (2007) noted above. However, we find that the proportion of total volatility due to the jump component is markedly larger during the Bear period than the combined and the Bull period.

The estimated parameters for the Bear period lead to negative skewness in both the risk neutral and the physical returns distribution, suggesting that the probability of a large decrease in stock prices exceeds the probability of a large increase. Jackwerth and Rubinstein (1996) termed this phenomenon as "crashophobia". The economic rationale for crashophobia is that put options are used as hedging instruments to protect against large downward movements in stock prices. This demand by investors due to portfolio insurance strategies has increased the price of protection (resulting in a "crash premium") and therefore the left tail of the risk neutral distribution has more weight.

Figure 2 presents the decomposed PBJD estimated distributions for the S&P 500 and NASDAQ using the MCMC parameter estimates for the Bear, Bull and the combined periods (Table 5). For both indexes, the figure show the crashophobia, as shown by the significant contribution of the down-jump component of the PBJD model.

For S&P 500's Bull period, we find $\lambda_d$ is comparable to $\lambda_u$. Whereas, both $\eta^{-1}$s are similar and smaller than the combined and the Bear period estimates. Consequently, the jump components together have a less significant impact on returns, effectively offsetting one another, and allowing $\mu$ to be the dominant force that pushes up the index level (see Figure 2). For NASDAQ's Bull period, $\lambda_u$ is larger than $\lambda_d$. That is the arrival of "good news" further amplifies the positive drift of this index (see Figure 2). Again, both $\eta^{-1}$s are similar and smaller than the combined and the Bear period estimates. The good news component accelerates the drift of the return process.

Mathematically, the PBJD and DEJD models are equivalent and, in theory, their parameter estimates should be nearly identical. For the simulated data, the parameter estimates in Table (1) are nearly identical. The parameter estimates for one stock (HYBD) in Table (4) are also comparable. However, the parameter estimates found in Table (3), the remainder of Table (4), and Table (5) are notably different from one another. The parameter estimates for the alternative TSJD models should be nearly identical when all the underlying distributional assumptions are satisfied (the independence of the returns, parameters, and jump events; their distributional forms; the hypothesis for the random generation of the data; and the equivalency of the mixture model components). In our simulation exercise, all of the above assumptions are satisfied and, as a result, the parameter estimates reported in Table (1) are very close to one another. Our analysis shows that the majority of estimated parameters are different under PBJD and DEJD. This divergence of the the parameter estimates indicates a violation of at least one of the underlying assumptions.

# 6 Conclusions

The paper develops a Markov Chain Monte Carlo (MCMC) technique for estimating the parameters of the Two-Sided Jump Diffusion models. We provide an empirical assessment of these models using daily returns for the S&P 500 and the NASDAQ indexes, as well as individual stocks. We complete our analysis by providing a comparison of the estimated parameters under MCMC and MLE. We find that, in general, the MCMC estimates are consistent with the MLE estimates. However, MCMC is computationally more efficient and yields smaller standard errors.

We find that the TSJD models are consistent with the empirical features of return processes. We study the behavior of the models' parameters during different market epochs(Bull and Bear periods) and find that with the introduction of the jump components, the volatility component due to the GBM part of the return process ($\sigma$) is constant across epochs. This finding is in contrast to the stochastic volatility models that show persistence in $\sigma$. As Eraker et al (2003) and others have shown, stochastic volatility is an important component of the return process and should be formally integrated into the TSJD specification. Then, one can formally test the conjecture that there is no persistence in $\sigma$, after adjustment for jumps. With the MCMC approach, it may be simpler to determine whether stochastic volatility remains important when the jump components of return process has a more complex structure like the TSJD specifications.

There are other interesting directions to extend of this work. As a starting point, other estimation techniques, such as the generalized method of moments and its variants may be utilized to obtain estimates of the TSJD parameters. Time-varying jump intensities, as proposed by Andersen et al (2002),offers another way to enhance the TSJD specification. Finally, integrating the jump detection techniques suggested by Lee and Hannig (2010) and Lee (2012) into the MCMC estimation of the TSJD represent a valuable but highly challenging direction for future research.

# 7 Appendix: The Details of the Proposed MCMC Estimation

**Sampling the Latent Variable**

Sampling the latent variable is the most important part of the MCMC approach. This sampling step is the "ownership" step of the algorithm, temporarily defining which mixture component of the model generates return. For each data point, we must identify if the data point is generated from the no-jump, up-jump, or down-jump trinomial mixture component. Let $I_i \in \{-1, 0, 1\}$ indicate a down, no, and an up-jump respectively for $i = 1, \ldots, M$. The conditional probability mass function of $I_i$ given $\theta'_{PBJD}$ and $r_i$ is:

$$\begin{aligned}
P\left(I_i = -1|\Psi', r_i\right) &= \frac{\lambda'_d f_{0,1}\left(r_i\right)}{\lambda'_d f_{0,1}\left(r_i\right) + \left(1 - \lambda'_u - \lambda'_d\right) f_{0,0}\left(r_i\right) + \lambda'_u f_{1,0}\left(r_i\right)} \\
P\left(I_i = 0|\Psi', r_i\right) &= \frac{\left(1 - \lambda'_u - \lambda'_d\right) f_{0,0}\left(r_i\right)}{\lambda'_d f_{0,1}\left(r_i\right) + \left(1 - \lambda'_u - \lambda'_d\right) f_{0,0}\left(r_i\right) + \lambda'_u f_{1,0}\left(r_i\right)} \\
P\left(I_i = 1|\Psi', r_i\right) &= \frac{\lambda'_u f_{1,0}\left(r_i\right)}{\lambda'_d f_{0,1}\left(r_i\right) + \left(1 - \lambda'_u - \lambda'_d\right) f_{0,0}\left(r_i\right) + \lambda'_u f_{1,0}\left(r_i\right)}
\end{aligned}$$

These give a proper probability mass function for $I_i$ which can be easily sampled from. Conditional on sampling an up-jump event $I_i = 1$ or down-jump event $I_i = -1$, we must also sample the size of the Pareto and Beta jump size. Let $Y_i^u$ and $Y_i^d$ be the conditional up- and down-jump sizes for return $r_i$.

Sampling $Y_i^u$ and $Y_i^d$ is a challenging step, because we are not able to draw samples from the posterior directly. The posterior distribution of the conditional jumps are:

$$\begin{aligned}
f\left(Y_i^u|\mu, \sigma^2, I_i = 1, \eta_u, r_i\right) &\propto e^{-\eta_u Y_i^u - \frac{1}{2\sigma^2}\left(r_i - Y_i^u - \mu + \frac{\sigma^2}{2}\right)^2} \\
f\left(Y_i^d|\mu, \sigma^2, I_i = -1, \eta_d, r_i\right) &\propto e^{\eta_d Y_i^d - \frac{1}{2\sigma^2}\left(r_i - Y_i^d - \mu + \frac{\sigma^2}{2}\right)^2}
\end{aligned}$$

Both are both non-standard, and it is not clear what the exact posterior distribution will be after dividing by a normalizing constant. However, it is a Metropolis-Hastings (MH) step to sample $Y_i^u$ and $Y_i^d$. Reasonable choices of proposal distributions are the Gamma and Normal distribution. We have found the Normal distribution to be superior, yielding the highest acceptance rates $(83 - 97\%)$ and the MH step is computationally intensive. In practice, we found it efficient and satisfactory to sample directly from the Normal distribution without the MH step. Sampling from the Normal proposal distribution without the accept-reject framework reduces the computational time significantly. Next, we sample the up- and down-jumps conditional on the jump indicators as follows:

- Conditional on $I_i = 1$, sample $Y_i^u$ from $N\left(r_i - \mu + \frac{\sigma^2}{2}, \sigma^2\right)$, forcing $Y_i^d = 0$.
- Conditional on $I_i = -1$, sample $Y_i^d$ from $N\left(r_i - \mu + \frac{\sigma^2}{2}, \sigma^2\right)$, forcing $Y_i^u = 0$.

Sampling the latent variable results in a *4-tuple* of complete information $R_i = \left(r_i, I_i, Y_i^d, Y_i^u\right)$ for $i = 1, \cdots, M$ forming $\underline{R} = (R_1, \cdots, R_M$. The remaining sampling steps for the elements of $\theta'_{PBJD}$ will have posterior distributions which rely on the following jump summary statistics, obtained from $R_i$:

$$U = \sum_{i=1}^{M} Y_i^u, n = \sum_{i=1}^{M} I\left(I_i = 1\right)$$

$$D = -\sum_{i=1}^{M} Y_i^d, m = \sum_{i=1}^{M} I\left(I_i = -1\right)$$

where, $U$ and $D$ are the sum of all of the sampled up- and down-jumps, and $n$ and $m$ are the number of sampled up- and down-jumps. The next component of the MCMC estimation is to sample the elements of $\theta_{PBJD}$ conditional on the complete data. Let $\underline{R}$ be the complete set of all *4-tuples*, $\underline{R} = (R_1, \cdots, R_M)$. The complete likelihood is:

$$\begin{aligned} L\left(\theta'_{PBJD}|\underline{R}\right) &\propto \prod_{\substack{i=1\\I_i=1}}^{M} \left\{\lambda'_u \eta_u e^{-\eta_u Y_i^u - \frac{1}{2\sigma^2}\left(r_i - Y_i^u - \mu + \frac{\sigma^2}{2}\right)^2}\right\} \\ &\times \prod_{\substack{i=1\\I_i=-1}}^{M} \left\{\lambda'_d \eta_d e^{\eta_d Y_i^d - \frac{1}{2\sigma^2}\left(r_i - Y_i^d - \mu + \frac{\sigma^2}{2}\right)^2}\right\} \\ &\times \prod_{\substack{i=1\\I_i=0}}^{M} \left\{\left(1 - \lambda'_u - \lambda'_d\right) e^{-\frac{1}{2\sigma^2}\left(r_i - \mu + \frac{\sigma^2}{2}\right)^2}\right\} \end{aligned}$$

The subscript in each product, $I_i$, indicates the product for up, down, and no-jump events. We use the complete likelihood to develop the framework to iteratively sample the parameters in $\theta'_{PBJD}$.

### Sampling the Jump Event Parameters

The arrival rate or intensity of the up- and down-events is governed by the Poisson processes with generating rates $\lambda_u$ and $\lambda_d$. We assume a vague prior $\pi\left(\lambda'_u, \lambda'_d\right) \propto I\left(\lambda'_u \in [0,1], \lambda'_d \in [0,1]\right)$, and only require information about the cumulative number of up and down-jumps which have occurred. This information in contained in the summary statistics of $\underline{R}$. The posterior sampling distribution of $(\lambda'_u, \lambda'_d)$ conditional on $\underline{R}$ is:

$$\begin{aligned} f\left(\lambda_u, \lambda_u|\underline{R}\right) &\propto \prod_{\substack{i=1\\I_i=1}}^{M} \lambda'_u \prod_{\substack{i=1\\I_i=-1}}^{M} \lambda'_d \prod_{\substack{i=1\\I_i=0}}^{M} \left(1 - \lambda'_u - \lambda'_d\right) \\ &\propto \left(\lambda'_u\right)^{n+1-1} \left(\lambda'_d\right)^{m+1-1} \left(1 - \lambda'_u - \lambda'_d\right)^{M-n-m+1-1} \end{aligned}$$

which is the kernel of a Dirichlet distribution, sample $(\lambda'_u, \lambda'_d) \sim Dirichlet\left(n+1, m+1, M-n-m+1\right)$.

### Sampling the Jump Size Parameters

The parameters that govern the generating process for the size of the up- and down-jumps are $\eta_u$ and $\eta_d$ respectively.

All of the information needed to develop the posterior distribution of $\eta_u$ is contained in the summary statistics of $\underline{R}$. The posterior distribution of $\eta_u$ is:

$$
\begin{aligned}
f(\eta_u|R) &\propto \prod_{\substack{i=1\\I_i=1}}^{M} \eta_u e^{-\eta_u Y_i^u - \frac{1}{2\sigma^2}(r_i - Y_i^u - \mu + \frac{\sigma^2 s}{2})^2} \\
&\propto \eta_u^{(n+1-1)} e^{-\eta_u U}
\end{aligned}
$$

which is the kernel of a Gamma distribution, sample $\eta_u \sim \Gamma(n+1, U)$. The posterior distribution of $\eta_d$ is similar, sample $\eta_d \sim \Gamma(m+1, D)$.

**Sampling the Drift Parameter**

The posterior distribution of parameter $\mu$ is comparable to the posterior distribution when the mixture model is the usual mixture of Normal distributions. In this case, terms involving $\eta_u$ and $\eta_d$ act as normalizing constants and do not contribute any information to the posterior distribution of $\mu$. The primary information which influences the posterior distribution of $\mu$ is contained in $\underline{R}$. The quadratic, Gaussian kernel of each component adjusts the observed $r_i$ by $Y_i^u$ or $Y_i^d$. Only one of the jump sizes will be non-zero by construction. So, we can re-write the posterior distribution to reflect this fact. The posterior distribution of $\mu$ is:

$$
\begin{aligned}
f\left(\mu|\underline{R}, \sigma^2\right) &\propto \prod_{\substack{i=1\\I_i=1}}^{M} e^{-\eta_u Y_i^u - \frac{1}{2\sigma^2}\left(r_i - Y_i^u - \mu + \frac{\sigma^2}{2}\right)^2} \\
&\times \prod_{\substack{i=1\\I_i=1}}^{M} e^{\eta_d Y_i^d - \frac{1}{2\sigma^2}\left(r_i - Y_i^d - \mu + \frac{\sigma^2}{2}\right)^2} \\
&\times \prod_{\substack{i=1\\I_i=0}}^{M} e^{-\frac{1}{2\sigma^2}\left(r_i - \mu + \frac{\sigma^2}{2}\right)^2} \\
&\propto e^{\frac{-1}{2\sigma^2}\sum_{i=1}^{M}\left(r_i - Y_i^u - Y_i^d - \mu + \frac{\sigma^2}{2}\right)^2}
\end{aligned}
$$

which is the kernel of a Normal distribution. Accordingly, sample $\mu \sim N\left(\frac{\sum_{i=1}^{M} r_i - Y_i^u - Y_i^d}{M}, \frac{\sigma^2}{M}\right)$.

**Sampling the Volatility Parameter**

The posterior distribution of $\sigma^2$ is:

$$f\left(\sigma^2|\underline{R},\mu\right) \propto \frac{1}{(\sigma^2)^{\left(\frac{M}{2}-1\right)+1}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{M}\left(r_i - Y_i^u - Y_i^d - \mu + \frac{\sigma^2}{2}\right)^2}$$

This is not non-standard distribution and we are unable to directly sample from the posterior. Subsequently, we rely on a MH accept-reject algorithm. However, the posterior closely resembles the kernel of an inverse Gamma distribution where the second parameter, located in the exponent actually depends on $\sigma^2$. Hence, we sample $\sigma^2$ by drawing from an inverse Gamma distribution. Then, we use the following MH accept-reject algorithm:

- Sample $\sigma^{2^*} \sim \Gamma^{-1}\left(\frac{M}{2} - 1, \sum_{i=1}^{M}\left(r_i - Y_i^u - Y_i^d - \mu + \frac{\sigma^2}{2}\right)^2\right)$
- Calculate the probability of accepting the candidate sample

  $$p^* = \frac{\pi\left(\sigma^{2^*}|\mu,\underline{R}\right)}{\pi\left(\sigma^2|\mu,\underline{R}\right)} \frac{\pi_{\Gamma^{-1}}\left(\sigma^2|\sigma^{2^*},\mu,\underline{R}\right)}{\pi_{\Gamma^{-1}}\left(\sigma^{2^*}|\sigma^2,\mu,\underline{R}\right)}$$

  where $\pi_{\Gamma^{-1}}()$ is the inverse Gamma distribution using the conditioned value of $\sigma^2$ or $\sigma^{2^*}$ in the construction of the exponent. Let $p^* = min\{p^*, 1\}$.

- Generate $p \sim Uniform(0,1)$ and if $p^* > p$ we accept $\sigma^{2^*}$.

Using simulated and real data, this MH schema for sampling $\sigma^2$ is efficient and accepts $80\% - 90\%$ of the candidate samples.

Table 1: **MCMC Parameter Estimates from Simulated Data.** The table presents parameter estimates under the PBJD and DEJD specifications using simulated data. Standard errors appear below the estimates.

| Model | $\lambda$ | $P$ | $\eta_u$ | $\eta_d$ | $\lambda_u$ | $\lambda_d$ | $\mu$ | $\sigma$ |
|---|---|---|---|---|---|---|---|---|
| PBJD-True | 0.3500 | 0.1429 | 10.000 | 10.000 | 0.0500 | 0.3000 | -0.0060 | 0.0200 |
| PBJD | 0.3543 | 0.1311 | 10.838 | 11.174 | 0.0464 | 0.3080 | -0.0061 | 0.0200 |
| | - | - | 0.0268 | 0.0118 | 0.0001 | 0.0004 | 0.0000 | 0.0000 |
| DEJD | 0.3649 | 0.1464 | 10.713 | 11.084 | 0.0534 | 0.3115 | -0.0062 | 0.0210 |
| | 0.001 | 0.000 | 0.026 | 0.011 | 0.000 | 0.000 | 0.000 | 0.000 |
| PBJD-True | 0.8750 | 0.4000 | 47.000 | 30.000 | 0.3500 | 0.0500 | -0.0060 | 0.0200 |
| PBJD | 0.8724 | 0.4171 | 44.368 | 28.047 | 0.3667 | 0.0504 | -0.0065 | 0.0194 |
| | - | - | 0.0944 | 0.1119 | 0.0020 | 0.0004 | 0.0000 | 0.0000 |
| DEJD | 0.8571 | 0.4413 | 44.136 | 27.740 | 0.3810 | 0.0603 | -0.0065 | 0.0195 |
| | 0.0011 | 0.0026 | 0.0927 | 0.1219 | - | - | 0.0000 | 0.0000 |
| DEJD-True | 0.3500 | 0.1429 | 10.000 | 10.000 | 0.0500 | 0.3000 | -0.0060 | 0.0200 |
| PBJD | 0.3156 | 0.1320 | 12.443 | 10.345 | 0.0417 | 0.2739 | -0.0069 | 0.0194 |
| | - | - | 0.0333 | 0.0100 | 0.0001 | 0.0003 | 0.0000 | 0.0000 |
| DEJD | 0.3293 | 0.1472 | 12.619 | 10.317 | 0.0486 | 0.2807 | -0.0069 | 0.0194 |
| | 0.0004 | 0.0005 | 0.0359 | 0.0106 | - | - | 0.0000 | 0.0000 |
| DEJD-True | 0.8750 | 0.4000 | 47.000 | 30.000 | 0.3500 | 0.0500 | -0.0060 | 0.0200 |
| PBJD | 0.9061 | 0.3978 | 44.135 | 31.071 | 0.3623 | 0.0355 | -0.0066 | 0.0200 |
| | - | - | 0.0920 | 0.1612 | 0.0019 | 0.0003 | 0.0000 | 0.0000 |
| DEJD | 0.8817 | 0.3933 | 42.482 | 31.516 | 0.3472 | 0.0461 | -0.0062 | 0.0201 |
| | 0.0011 | 0.0031 | 0.1075 | 0.1965 | - | - | 0.0000 | 0.0000 |

Table 2: **Sample statistics for indexes and equities.** The table present sample moments for daily returns for S&P 500, NASDAQ, and five individual stocks. The date range for the indexes appear in the table. The daily returns for the stocks span the period 1/1/1999 through 12/31/2003 ($N = 1256$).

| Name | Date | Minimum | Median | Maximum | Mean | SD | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|---|
| S&P 500 | 7/62-12/03 | -0.2047 | 0.0004 | 0.0910 | 0.0003 | 0.0095 | -0.9448 | 25.758 |
| | 05/07-12/10 | -0.0947 | 0.0008 | 0.1096 | -0.0002 | 0.0180 | -0.1758 | 9.185 |
| | 05/07-03/09 | -0.0947 | 0.0001 | 0.1096 | -0.0017 | 0.0218 | -0.1311 | 7.800 |
| | 03/09-12/10 | -0.0437 | 0.0013 | 0.0684 | 0.0013 | 0.0132 | 0.2934 | 5.867 |
| NASDAQ | 1/73-12/03 | -0.1132 | 0.0011 | 0.1427 | 0.0005 | 0.0124 | -0.0823 | 10.778 |
| | 05/07-12/10 | -0.0959 | 0.0011 | 0.1116 | 0.0001 | 0.0185 | -0.1261 | 7.731 |
| | 05/07-03/09 | -0.0959 | -0.0004 | 0.1116 | -0.0015 | 0.0220 | -0.0883 | 6.745 |
| | 03/09-12/10 | -0.0419 | 0.0016 | 0.0683 | 0.0016 | 0.0140 | 0.2610 | 5.4823 |
| | Ticker | | | | | | | |
| AT & T CP | T | -0.1908 | -0.0022 | 0.2317 | -0.0006 | 0.0296 | 0.4847 | 5.732 |
| HYCOR BIO | HYBD | -0.4590 | 0.0000 | 0.5852 | 0.0033 | 0.0680 | 1.3077 | 10.906 |
| INTEL CP | INTC | -0.2203 | -0.0004 | 0.2012 | 0.0007 | 0.0360 | -0.0916 | 3.120 |
| LIFECORE BIO | LCMB | -0.5754 | 0.0000 | 0.5387 | 0.0009 | 0.0499 | -0.1043 | 30.443 |
| MFRI CP | MFRI | -0.2264 | 0.0000 | 0.5044 | 0.0009 | 0.0534 | 1.4315 | 12.678 |

Table 3: **Comparison of MCMC and MLE for Indexes.** The table presents estimates for S&P 500 and NASDAQ indexes under the PBJD and DEJD specifications. The daily returns for S&P 500 spans the period 7/1/1962-12/31/2003 ($N = 10466$). The daily returns for NASDAQ spans the period 1/1/1973-12/31/2003 ($N = 7828$). The parameters $\lambda_u$ and $\lambda_d$ under the DEJD are calculated values implied by the estimates of the mixing parameter $p$ and $\lambda$. Standard errors appear below the estimates. All estimates are statistically significant at 99% and higher.

| Symbol | Model - Method | $p$ | $\lambda$ | $\eta_u$ | $\eta_d$ | $\lambda_u$ | $\lambda_d$ | $\mu$ | $\sigma$ |
|---|---|---|---|---|---|---|---|---|---|
| S&P 500 | PBJD - MLE | 0.4521 | 1.0438 | 173.91 | 185.98 | 0.4719 | 0.5719 | 0.0008 | 0.0046 |
| | | - | - | 0.43 | 0.44 | 0.0714 | 0.0933 | 0.0000 | 0.0000 |
| | PBJD - MCMC | 0.4055 | 1.0348 | 151.51 | 151.85 | 0.4196 | 0.6152 | 0.0010 | 0.0040 |
| | | - | - | 0.0961 | 0.0857 | 0.0007 | 0.0017 | 0.0000 | 0.0000 |
| | DEJD - MCMC | 0.4294 | 1.6931 | 152.01 | 151.83 | 0.7270 | 0.9661 | 0.0011 | 0.0039 |
| | | 0.0009 | 0.0060 | 0.0888 | 0.0905 | - | - | 0.0000 | 0.0000 |
| NASDAQ | PBJD - MLE | 0.3455 | 0.6674 | 95.90 | 110.38 | 0.2306 | 0.4368 | 0.0021 | 0.0050 |
| | | - | - | 0.60 | 0.70 | 0.0238 | 0.0352 | 0.0000 | 0.0000 |
| | PBJD - MCMC | 0.3225 | 0.6004 | 89.38 | 90.75 | 0.1936 | 0.4068 | 0.0021 | 0.0050 |
| | | - | - | 0.0766 | 0.0541 | 0.0003 | 0.0005 | 0.0000 | 0.0000 |
| | DEJD - MCMC | 0.3487 | 2.0214 | 77.751 | 70.560 | 0.7049 | 1.3165 | 0.0037 | 0.0062 |
| | | 0.0017 | 0.0182 | 0.1691 | 0.1090 | - | - | 0.0000 | 0.0000 |

Figure 1: PBJD distribution for S&P 500 (7/1/1962-12/31/2003)
using the MCMC parameter estimates (Table 3)

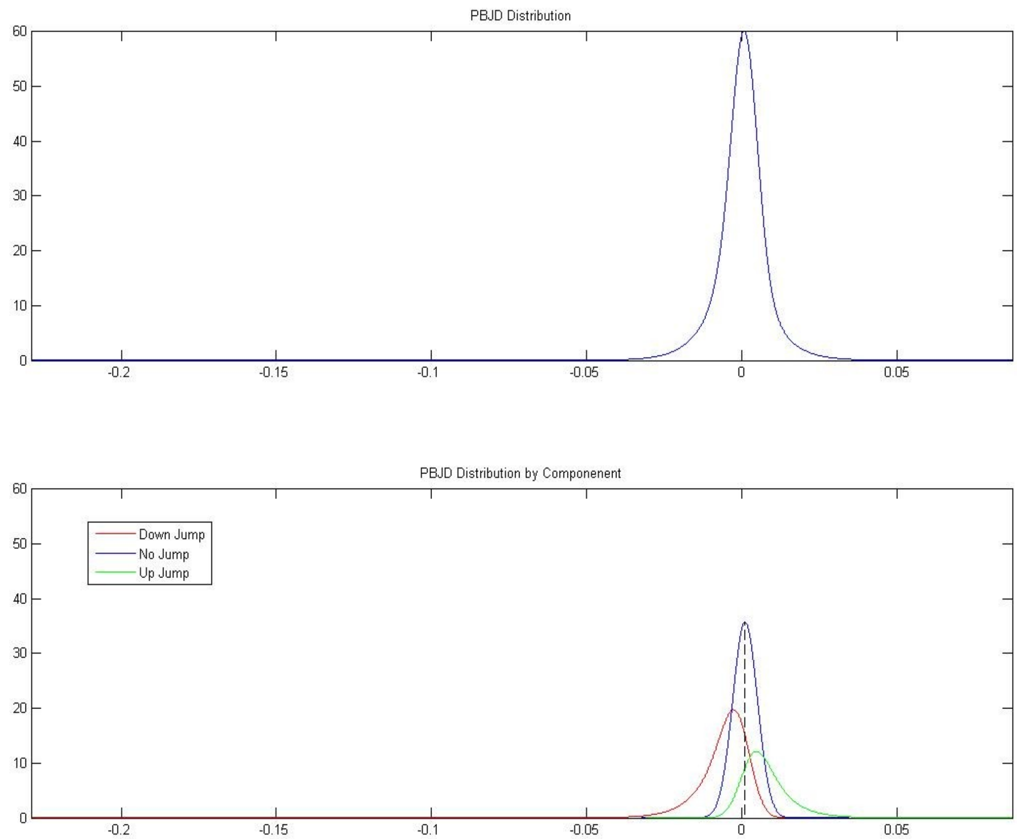


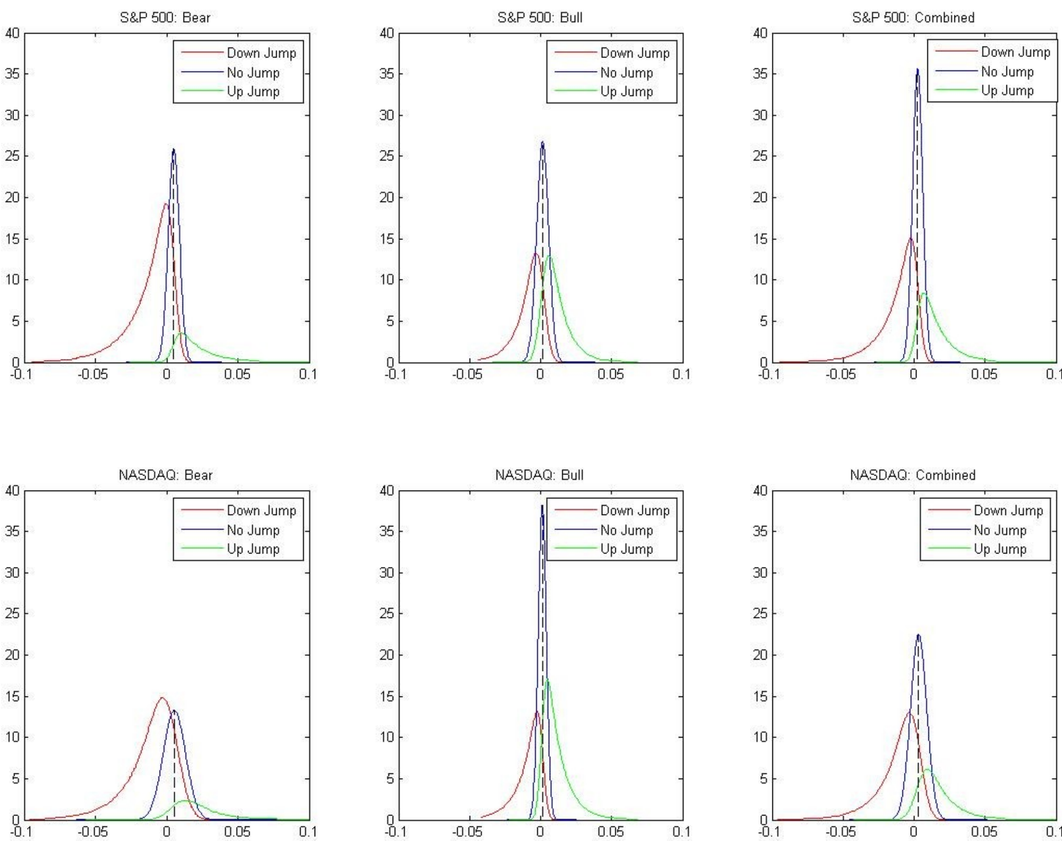
Table 4: **Comparison of MCMC and MLE for Individual Stocks.** The table presents estimates for nine stocks under the PBJD and DEJD specifications. The daily returns span the period 1/1/1999-12/31/2003 ($N = 1,256$). The parameters $\lambda_u$ and $\lambda_d$ under the DEJD are calculated values implied by the estimates of the mixing parameter $p$ and $\lambda$. Standard errors appear below the estimates.

| Symbol | Model - Method | $p$ | $\lambda$ | $\eta_u$ | $\eta_d$ | $\lambda_u$ | $\lambda_d$ | $\mu$ | $\sigma$ |
|---|---|---|---|---|---|---|---|---|---|
| **T** | PBJD - MLE | 0.8755 | 0.3969 | 47.35 | 28.97 | 0.3475 | 0.0494 | -0.0059 | 0.0210 |
| | | - | - | 2.42 | 2.15 | 0.1199 | 0.1225 | 0.0011 | 0.0000 |
| | PBJD - MCMC | 0.8508 | 0.8594 | 46.98 | 43.88 | 0.7312 | 0.1282 | -0.0085 | 0.0190 |
| | | - | - | 0.12 | 0.24 | 0.0053 | 0.0018 | 0.0000 | 0.0000 |
| | DEJD - MCMC | 0.8028 | 0.5645 | 41.42 | 37.27 | 0.4532 | 0.1113 | -0.0061 | 0.0206 |
| | | 0.0018 | 0.0059 | 0.11 | 0.22 | - | - | 0.0000 | 0.0000 |
| **HYBD** | PBJD - MLE | 0.3921 | 1.1884 | 19.41 | 33.25 | 0.4660 | 0.7224 | -0.0013 | 0.0230 |
| | | - | - | 0.42 | 0.41 | 0.2365 | 0.3368 | 0.0003 | 0.0000 |
| | PBJD - MCMC | 0.4585 | 1.1312 | 18.54 | 24.60 | 0.5186 | 0.6126 | 0.0003 | 0.0131 |
| | | - | - | 0.03 | 0.04 | 0.0016 | 0.0019 | 0.0000 | 0.0000 |
| | DEJD - MCMC | 0.4536 | 1.4258 | 16.97 | 25.05 | 0.6467 | 0.7791 | 0.0002 | 0.0218 |
| | | 0.0010 | 0.0087 | 0.03 | 0.04 | - | - | 0.0001 | 0.0000 |
| **INTC** | PBJD - MLE | 0.8475 | 0.4000 | 47.22 | 24.49 | 0.3390 | 0.0610 | -0.0036 | 0.0281 |
| | | - | - | 1.88 | 1.69 | 0.1590 | 0.1541 | 0.0016 | 0.0000 |
| | PBJD - MCMC | 0.5176 | 0.8357 | 47.85 | 49.39 | 0.4326 | 0.4031 | 0.0002 | 0.0142 |
| | | - | - | 0.06 | 0.06 | 0.0017 | 0.0013 | 0.0000 | 0.0000 |
| | DEJD - MCMC | 0.4858 | 1.1026 | 46.69 | 49.34 | 0.5356 | 0.5670 | 0.0005 | 0.0144 |
| | | 0.0746 | 0.0041 | 0.06 | 0.07 | - | - | 0.0000 | 0.0000 |
| **LCBM** | PBJD - MLE | 0.7051 | 0.4276 | 24.22 | 20.57 | 0.3015 | 0.1261 | -0.0051 | 0.0234 |
| | | - | - | 1.77 | 1.56 | 0.0613 | 0.0735 | 0.0010 | 0.0000 |
| | PBJD - MCMC | 0.4464 | 1.1437 | 31.73 | 39.18 | 0.5105 | 0.6332 | 0.0005 | 0.0077 |
| | | - | - | 0.05 | 0.07 | 0.0017 | 0.0025 | 0.0000 | 0.0000 |
| | DEJD - MCMC | 0.4705 | 1.5425 | 29.75 | 37.92 | 0.7257 | 0.8168 | 0.0000 | 0.0109 |
| | | 0.0008 | 0.0089 | 0.04 | 0.06 | - | - | 0.0000 | 0.0000 |
| **MFRI** | PBJD - MLE | 0.4885 | 1.4696 | 33.07 | 35.67 | 0.7179 | 0.7517 | 0.0002 | 0.0038 |
| | | - | - | 0.11 | 0.15 | 0.1556 | 0.1882 | 0.0000 | 0.0000 |
| | PBJD - MCMC | 0.4985 | 1.0921 | 29.26 | 32.03 | 0.5444 | 0.5477 | 0.0002 | 0.0065 |
| | | - | - | 0.06 | 0.05 | 0.0019 | 0.0016 | 0.0000 | 0.0000 |
| | DEJD - MCMC | 0.4984 | 1.9761 | 29.28 | 32.00 | 0.9849 | 0.9912 | 0.0002 | 0.0064 |
| | | 0.0004 | 0.0079 | 0.06 | 0.01 | - | - | 0.0000 | 0.0000 |

Table 5: **MCMC Parameter Estimates for the Indexes.** The table presents estimates for S&P 500 and NASDAQ indexes under the PBJD and DEJD specifications using recent data (5/1/2007 through 12/31/2010). The parameters $\lambda_u$ and $\lambda_d$ under the DEJD are calculated values implied by the estimates of the mixing parameter $p$ and $\lambda$. Standard errors appear below the estimates.

| Symbol | Model | $p$ | $\lambda$ | $\eta_u$ | $\eta_d$ | $\lambda_u$ | $\lambda_d$ | $\mu$ | $\sigma$ |
|---|---|---|---|---|---|---|---|---|---|
| S&P 500 | PBJD | 0.3784 | 1.1489 | 80.67 | 72.16 | 0.4348 | 0.7141 | 0.0025 | 0.0035 |
| 05/2007-12/2010 | | - | - | 0.12 | 0.08 | 0.0020 | 0.0025 | 0.0000 | 0.0000 |
| | DEJD | 0.4087 | 1.6977 | 78.75 | 70.88 | 0.6938 | 1.0039 | 0.0024 | 0.0042 |
| | | 0.0008 | 0.0054 | 0.12 | 0.08 | - | - | 0.0000 | 0.0000 |
| S&P 500 | PBJD | 0.2159 | 1.3805 | 63.69 | 60.11 | 0.2980 | 1.0825 | 0.0050 | 0.0039 |
| 05/2007-03/2009 | | - | - | 0.15 | 0.08 | 0.0011 | 0.0047 | 0.0000 | 0.0000 |
| | DEJD | 0.2769 | 2.8641 | 61.94 | 60.54 | 0.7931 | 2.0710 | 0.0024 | 0.0042 |
| | | 0.0015 | 0.0242 | 0.18 | 0.11 | - | - | 0.0000 | 0.0000 |
| S&P 500 | PBJD | 0.4939 | 1.2635 | 104.84 | 103.03 | 0.6241 | 0.6394 | 0.0015 | 0.0042 |
| 03/2009-12/2010 | | - | - | 0.25 | 0.25 | 0.0046 | 0.0057 | 0.0000 | 0.0000 |
| | DEJD | 0.4803 | 3.0172 | 102.69 | 102.35 | 1.4732 | 1.5940 | 0.0017 | 0.0031 |
| | | 0.0021 | 0.0242 | 0.17 | 0.17 | - | - | 0.0000 | 0.0000 |
| NASDAQ | PBJD | 0.3514 | 1.1352 | 76.75 | 70.63 | 0.3989 | 0.7363 | 0.0032 | 0.0057 |
| 05/2007-12/2010 | | - | - | 0.17 | 0.09 | 0.0025 | 0.0031 | 0.0000 | 0.0000 |
| | DEJD | 0.3499 | 1.9247 | 74.93 | 70.38 | 0.6735 | 1.2512 | 0.0038 | 0.0059 |
| | | 0.0013 | 0.0140 | 0.16 | 0.10 | - | - | 0.0000 | 0.0000 |
| NASDAQ | PBJD | 0.1999 | 1.3374 | 61.58 | 60.55 | 0.2673 | 1.0701 | 0.0055 | 0.0079 |
| 05/2007-03/2009 | | - | - | 0.19 | 0.16 | 0.0018 | 0.0063 | 0.0001 | 0.0000 |
| | DEJD | 0.2720 | 2.2792 | 60.00 | 62.29 | 0.6199 | 1.6593 | 0.0050 | 0.0085 |
| | | 0.0016 | 0.0202 | 0.12 | 0.08 | - | - | 0.0001 | 0.0000 |
| NASDAQ | PBJD | 0.5428 | 1.2989 | 99.80 | 96.46 | 0.7051 | 0.5938 | 0.0012 | 0.0029 |
| 03/2009-12/2010 | | - | - | 0.16 | 0.16 | 0.0033 | 0.0027 | 0.0000 | 0.0000 |
| | DEJD | 0.5674 | 3.7589 | 99.47 | 95.19 | 2.1328 | 1.6261 | 0.0004 | 0.0016 |
| | | 0.0010 | 0.0217 | 0.12 | 0.13 | - | - | 0.0000 | 0.0000 |

Figure 2: PBJD distributions for S&P 500 and NASDAQ (Bull, Bear, and Combined periods) using the MCMC parameter estimates (Table 5)



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