A NON-PARAMETRIC APPROACH TO PRICING FINANCIAL ASSETS *

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ABSTRACT

Asset valuation models are among the most important contribution of finance to economic theory. Two distinct approaches to asset pricing have a long tradition in this literature and are widely used by practitioners. The first is based on the notion of *arbitrage*, the idea that in an efficient financial market the value of an asset must at least equal the combined value of its constituent components. A violation would create opportunities for rational-competitive investors to create something out of nothing, a *free lunch*. Hence, in an information-ally efficient competitive economy, assets prices are determined in a manner that rules out both static and inter temporal arbitrage. This approach suggests a linear functional relating any asset's returns to to returns on *unidentified* basis assets (factors).

The second approach asserts that asset prices are determined by the demand and supply conditions in the asset markets. Asset demand functions emerge from optimizing portfolio behavior by atomistic investors who take the supply of assets as a given. A variety of equilibrium asset pricing relationships can then be derived but their functional form will be dependent upon the specification of individuals' objective function (preferences). This approach suggests specific factors that have a bearing on asset price determination but provides no guidance on how to select among the competing *parametric* functional forms.

The purpose of the present research is to develop a new methodology that circumvents the problems associated with both approaches. A general portfolio choice model that nests both arbitrage and equilibrium pricing relationships is proposed. The model allows an unspecified number of factors to enter asset pricing relationships, and permits a wide class of functional specifications. It is argued that portfolio theory has limited bearing on the identification of the correct pricing factors and the selection of the appropriate functional form – these tasks are generally statistical in nature. Non-parametric functional form estimation techniques and algorithms for selecting pricing factors are proposed to support this contention. Statistical tests to measure the performance of the new model relative to existing models are proposed. Preliminary analysis suggest that the new procedures are superior to existing alternatives.

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At times, the parametric approach [to asset pricing] looks like a fishing expedition without a well-articulated strategy for finding the promising fishing holes[1]

Asset valuation models are among the most important contribution of finance to economic theory. Two distinct approaches to asset pricing have a long tradition in this literature and are widely used by practitioners. The first is based on the notion of *arbitrage*, the idea that in an efficient financial market the value of an asset must at least equal the combined value of its constituent components. A violation would create opportunities for rational-competitive investors to create something out of nothing, a *free lunch.* Hence, in an information ally efficient competitive economy, assets prices are determined in a manner that rules out both static and inter-temporal arbitrage. This approach suggests a linear functional relating any asset's returns to to returns on *unidentified* basis assets (factors).

The second approach asserts that asset prices are determined by the demand and supply conditions in the asset markets. Asset demand functions emerge from optimizing portfolio behavior by atomistic investors who take the supply of assets as a given. A variety of equilibrium asset pricing relationships can then be derived but their functional form will be dependent upon the specification of individuals' objective functions (preferences). This approach suggests specific factors that have a bearing on asset price determination but provides no guidance on how to select among the competing *parametric* functional forms.

This paper develops a new model that circumvents the problems associated with both approaches. In section (I) a general portfolio choice model that nests both arbitrage and equilibrium pricing relationships as its special case is proposed. The model allows an unspecified number of factors to enter asset pricing relationships, and permits a wide class of functional specifications. It is shown that market imperfections such as transactions costs, short sale and borrowing restrictions, and taxation introduce severe nonlinearities into the pricing relationships derived from a very general portfolio choice models. In section (II) it is argued that portfolio theory has limited bearing on the identification of the correct pricing factors and the selection of the appropriate functional form – these tasks are generally statistical in nature. Non-parametric functional form estimation techniques and algorithms for selecting pricing factors are proposed to support this contention. Statistical tests to measure the performance of the new methodology relative to existing models are also developed. Preliminary analysis suggest that the new procedures are superior to existing alternatives. In section (III) data, estimation procedures, and preliminary findings are discussed.

MODEL:

A large body of empirical evidence in finance literature indicate that existing asset pricing models are grossly inconsistent with actual data. The failure is often attributed to unspecified "factors" for the arbitrage based models and to unspecified "functional form" in the case of preference based models. In this section a general portfolio choice model that presumes no specific objective function is proposed. The objective function is dependent upon generic factors – qualitative attributes – that determine any asset's value. It is assumed that an asset's value is determined by its "quality". How to define and measure quality is a subjective matter that varies across investors and is further discussed below.

In the economic literature product quality has been viewed in two ways: First, the Generalized Commodity approach developed by Houthakker (1951) and Theil (1952). Second, the Differentiated Commodity due to Lancaster (1966). A related distinction has been drawn between a discrete versus a continuous spectrum of product quality as well, see for example Hanemann (1980). In the existing portfolio choice models assets are viewed as a generalized commodity with their rate of return as the single quality indicator that has a continuous spectrum of values.

To understand the influence of the qualitative attributes of assets on portfolio choice behavior a model which combines all the above features is constructed and testable hypothesis are derived. The model accounts for market imperfections that result in nonlinear budget constraint faced by individuals. These include nonlinear pricing schemes, transactions costs, taxes, and institutional limitation such as borrowing and short sale constraints. In it is shown that the proposed model is consistent with all existing asset pricing models and therefore nests them as special case.

The model proposed is an extension of the authors previous research (Ramezani 1991). The salient feature of the proposed model is that investors are viewed as producers of portfolios with non-market characteristics (similar to the household production theory of Becker (1965) and Muth (1966)). Individuals are assumed to derive utility from consumption and the attributes of portfolios they hold. Portfolio attributes influence utility because they provide services (e.g., a sense of social responsibility, liquidity, etc.) and they enhance investors ability to smooth inter-temporal income and therefore consumption.

This characterization of investment behavior is based on the observation that individuals combine marketed assets, which may include their own labor and human capital, to *produce* utility-bearing non-marketed portfolio attributes. This characterization is consistent with the existing models put forth by Markowitz (1952), Sharpe (1964), Rubinstein (1973), Ingersoll (1975), Kraus & Litzenberger (1976), and Litzenberger & Ronn (1986). However, the proposed model provides additional insights into investment behavior. A brief sketch of the model follows. Detailed presentation can be found in Ramezani (1991).

The Asset-Attribute Transformation Frontier: The first focus in developing the proposed model is on establishing the technical relationship between assets and *portfolio attributes*. The representation of the process of creating portfolio attributes parallels the production theory of the firm (Debreu 1959) with the exception of three distinctions. First, 'production' is undertaken by investors who face constraints that are different than those faced by firms. Second, the possibility of short sales implies that assets as 'inputs' to attribute production could take on negative values. Third, portfolio attributes result from joint production function where an attribute will influence the availability of other attributes (e.g., liquid assets have lower transactions costs).

Let $X \in R^n$ be vector of marketed assets (e.g., stocks, bonds, 'market portfolio', risk free asset, etc.) and $x \in X$ a subset used to form a portfolio. The following assumptions are made with regard to existing assets:

A1: Assets may be characterized by r possible quality attributes. $r^* \subseteq r$ are assumed to be common to all assets. The remainder are 'unique'. Also, $r^* < n$ (n is the number of marketed assets).

A2: Quality parameters are denoted by $\beta \in R^s$; $b_{ij} \in \beta_i$ is the amount of attribute j in a unit of asset i. Each asset has at least one unique attribute. The vector β is exogenous to investors. The dimension of attribute space β can vary across markets and is determined by the following inequality:

$$n \times (r^* + 1) \leq s \leq r^* \times (n - 1) + r$$

A3: $Z \in R^m$ Utility bearing attributes, are related to X via a joint transformation function $G(X, Z; \beta) \leq 0$, i.e. a mapping from R^s into R^m, which explicitly depends on the vector β. The limits of m are: (a single asset) $r^* + 1 \leq m \leq r$ (all assets or the 'market portfolio')

A4: $G(X, Z; \beta)$ is monotonic and convex in Z and X. Monotonicity implies that we can represent a portfolio's k^{th} attribute as a function of its constituent assets and other attributes, i.e., $z_k = G_k(X, z_{m-k}; \beta)$. Also, holding all other attributes (z_{m-k}) constant, it is assumed that $G_k(..)$ is a quasiconcave function of X.

When the level of an attribute generated by a portfolio is independent of other attributes obtained from the same portfolio, the transformation functions is said to be *separable*; $z_k = G_k(X; \beta_k)$. The structure proposed so far is most general and very flexible in terms of covering a variety of possibilities. Next the preferences of a 'representative' individual for portfolio attributes are described.

The Nature of Preferences

A5: Individual preferences over $Z \in R^m$ is representable by a continuous, real-valued utility function, $u : R^m \to R$. Attributes are measured in such way that marginal utility of all attributes is positive.

Individual Budget Constraints

A6: Individuals face nonlinear budget constraint as a result of market imperfections (e.g., taxes and transaction costs) and institutional rules (short sale limits and borrowing constraints). It is assumed that this nonlinearity can be represented as $P = \Theta(Q)$, where Q is the vector of actual asset prices prior to commissions and taxes, and P is the vector of 'effective' prices faced by the individual after correcting for such costs. Furthermore, $\Theta(.)$ is assumed to be a vector of monotonic functions, $\theta_i(.)$, each relating the effective price of individual asset i, p_i, to its market price q_i. This implies individual's budget constraints will be of the form $P'x \leq W$, where W is individual's wealth. In the absence of market imperfection effective and actual asset prices will be identical.

Deriving Qualitative Results

The theoretical structure outlined above provides two means of obtaining testable hypotheses from the model:

Utility Maximization: Investors choose $x \in X$ to maximize $u(Z)$ subject to $G(x, Z, \beta) \leq 0$ and $P'x \leq W$, where P is the vector of 'effective' asset prices and W is individual's wealth and short sales are allowed.

Proposition 1 : There exist a set of n quality augmented asset demand functions $X = X(P, W, \beta)$, m attribute demand functions $Z = Z(P, W, \beta)$, an indirect utility function $V = V(P, W, \beta)$ and a set of price decomposition equations such that

$$p_i = \sum_{k=1}^{m} \lambda_k(P, Z, \beta)[\partial G_k/\partial x_i]. \tag{1}$$

Proof: The first order necessary conditions for the optimization problem, choose x so as to

$$\text{Max} \;\; u(Z) \quad \text{subject to} \;\; G(X, Z, \beta) \;\leq\; 0 \;, \quad \text{and} \;\; P'x \;\leq\; W \quad \text{are :}$$

$$\sum_{k=1}^{m} \frac{\partial u}{\partial z_k} \frac{\partial G_k}{\partial x_i} - \lambda \; p_i \;\; = \;\; 0$$

Given the assumptions on $u(\;.\;)$ and $G(\;.\;)$, the First Order Necessary Conditions can be solved for asset demands functions $X(\;.\;)$.[2] Substituting these into $G(\;.\;)$, the optimum level of attributes $Z(\;.\;)$ may be expressed as a function of wealth, prices, and the quality parameters. Substituting Z into $u(\;.\;)$ the indirect utility function is obtained. Solving the first order conditions for p_i and utilizing the definition $\lambda = \partial u / \partial W$ gives the price decomposition equation where $\lambda_k = \partial W / \partial z_k$ is the implicit value or 'shadow value' of the k^{th} attribute.

Equation (1) provides a method of linking asset prices and their attributes and is the basis for the empirical part of this paper. To emphasize the nonlinear nature of this relationship we express equation (1) in terms of the actual asset prices:

$$\theta_i(q_i) = \sum_{k=1}^{m} \lambda_k(P, Z, \beta)[\partial G_k / \partial x_i]. \tag{2}$$

The Dual Approach: The dual to the utility maximization is itself a two stage optimization problem:

- Stage 1: Investor's aim is to minimize the cost of achieving a vector of attributes subject to $G(.) \leq 0$: Choose x to min $P'x$ given $G(.)$, this generates the assets-attributes efficient frontier.

- Stage 2: Utility is maximized subject to the optimum *cost function* : choose Z so as to *max* $u(Z)$ subject to $E(P, Z; \beta) = W$.

The optimum portfolio is at the tangency of the indifference surface and the cost efficient frontier. The two stage optimization methodology leads to the second method of obtaining testable hypothesis from this model.

Proposition 2: Given $G(\;.\;)$ there exists an expenditure function $E(P, Z; \beta)$ such that $\partial \; E(\;.\;) \; / \; \partial \; z_k \;\; = \;\; \Gamma_k(P, Z, \beta)$ and $\partial \; E(\;.\;) \; / \; \partial \; p_i = x_i(P, Z, \beta)$.

Proof: See page 30 of Ramezani (1991).

This second approach provides new insights on the analysis of capital market efficiency. Appendix A shows that the preceding theoretical structure is sufficiently general to subsume preference based static and dynamic asset pricing, as well as arbitrage models, as special case and therefore facilitate a 'nested hypothesis test' of various alternative. In the next section a non-parametric procedure is proposed that will uncover the functional form of the asset pricing relationship depicted in equation (2) for a given vector of actual asset prices and assets attributes. Procedures for selecting the attributes that enter the pricing relationship are discussed in section 4.

Non-parametric Functional Form Estimation

There are various occasions when economic theory, the data, or both suggest a nonlinear relationship between the dependent and the explanatory variables under consideration. Consumer and production theory provide several examples. Asset valuation relationship, as shown above, provide an example of severe nonlinear interaction that are driven both by preferences and market imperfections.

The specification of a functional form is not a straightforward decision and, as is evident in applied economic literature, different functional forms are selected on the basis of their tractability rather than a priori knowledge of the true functional relation. The usual practice is to take the data as given and to impose a structure that is sufficiently general, as with flexible functional forms, or to search for the appropriate structure within a narrow class of specifications, as with the Box-Cox transformation. These methods impose structure upon the data.

An alternative approach, advanced by non-parametric techniques, is to let the data determine the best specification. Non-parametric methods of functional form estimation offer the means to avoid the problems associated with arbitrary specification. This is particularly true of procedures that invoke few stringent distributional assumptions. Combined with other appealing features, this flexibility has resulted in the popularity of non-parametric methods, and due to advances in computing, these procedures have become more accessible (Efron & Tibshirani 1991).[3]

This section introduces a specific non-parametric curve estimation technique called the Additivity and Variance Stabilization (AVAS) method (Tibshirani 1988). Given space limitations, the presentation will be brief, emphasizing the intuition and usefulness of AVAS rather than technical details, for which the interested readers should consult Appendix B and Tibshirani (1988). The purpose of this section is simply to elucidate the potential utility of these methods in uncovering nonlinearities in asset pricing models. As shown, such an exercise is helpful for specifying a parametric model that does not impose an ad hoc specification and is relatively simple to interpret.

The AVAS belongs to a broad class of functional specifications called the Generalized Additive Models (GAM), discussed in detail by Hastie & Tibshirani (1990). As will become apparent, the additive specification in general and AVAS in particular are well suited to the analysis of economic data because they admit nonlinearity of both the *dependent* and the *explanatory* variables and accommodate interaction effects. Under the GAM specification, an arbitrary function of the dependent variable is related to the sum of arbitrary functions of the independent variable(s). The approximation provided by these arbitrary functions and the additive structure is superior to the linear specification. The costs of gaining these flexibilities are in terms of statistical inference, the added modeling and interpretive efforts required, and the computational costs of implementing these techniques.

The AVAS specification assumes that an arbitrary function of the dependent variable, $\Theta(D)$, is related to functions of the independent variables, $F_i(I_i)$, via an additive structure of the form:

$$\Theta(D) \ = \sum_{i=1}^{n} \ F_i(I_i) \ + \epsilon \tag{3}$$

where the subscript i refers to the i-th explanatory variable, and it is assumed that $\Theta(D)$ is monotone and strictly increasing, $F_i(I_i)$'s have a multivariate normal distribution, ϵ is normally distributed with mean zero, and ϵ is independent of I_i's. Both the dependent and the explanatory variables may be categorical or continuous. An additional advantage, as Tibshirani (1988) has shown, is that AVAS is a generalization of the Box-Cox transformation. AVAS is, however, superior to Box-Cox in the sense that the transformations it provides are not limited to the logarithmic class of functions (Hastie & Tibshirani 1990, page 187).

The specification in (3) was first proposed by Breiman & Friedman (1985) in their Alternating Conditional Expectations (ACE) model. AVAS is similar to ACE in every respect except that functions $\Theta(D)$ and $F_i(I_i)$'s are chosen to achieve constant residual variance. Tibshirani (1988) and Hastie & Tibshirani (1990, page 194) show that AVAS improves upon ACE in that it does not lead to erroneous results in the presence of outliers or the violations of the distributional assumptions. They also show that AVAS is more appropriate for regression while ACE is most useful for correlation analysis. The key point to note is that given the GAM formulation in (3), the aim for any procedure – ACE, AVAS, others – is to obtain non-parametric estimates of the *functions* $\Theta(D)$ and $F_i(I_i)$'s for any given data.

Note that despite its additive structure, the formulation in equation (3) is quite general: each term can be a complex or linear function, and a function of more than one explanatory variable. Thus $F_k(I_k)$, where $S_k = I_i * I_j$ and $F_k(\ .\ )$ is an unspecified function, may be a term in the model. At first glance, the underlying distributional assumptions may appear stringent. However, when viewed in the context of economic data, these assumptions are indeed appropriate. For example, AVAS permits the explanatory variables to be jointly distributed, which is desirable in the case of most economic data, particularly for asset attributes. As for normality, the reasonableness of this assumption may only be assessed given the size and the context of the specific data analyzed. The AVAS criteria for selecting the appropriate functions for the dependent and explanatory variables and other related issues are described in Appendix A. The remainder of this section is devoted to the application of AVAS to the asset pricing model proposed above.

Application to Asset Pricing:

Comparison of the AVAS model (3) with the general asset pricing relationship (2) shows important similarities. The starting point of any empirical analysis is equation (2):

$$\theta_i(q_i) = \sum_{k=1}^{m} \lambda_k(P, Z, \beta)[\partial G_k / \partial x_i]. \tag{2}$$

Further simplification is required before AVAS can be applied to (2). In particular it is assumed that the transformation functions for portfolio attributes are separable and linear, i.e., $z_k = \sum_{i=1}^{n} b_{ik} x_i$ or equivalently $b_{ik} = [\partial G_k / \partial x_i]$. Substituting back into (2) we obtain:

$$\theta_i(q_i) = \sum_{k=1}^{m} \lambda_k(P, Z, \beta) b_{ik} = \sum_{i=1}^{n} \ F_i(b_{ik}) \ + \epsilon \tag{4}$$

where $F_i(b_{ik}; P, Z, \beta)$ shows the functional relation of asset prices to their attributes for given vectors of P and Z, and ϵ is the error term defined as above. Equation (4) will be the starting point for the non-parametric analysis. Both time series and cross-section data may be utilized. In the former case changes in the price of a particular asset would be related to changes in its attribute over time. The latter case is of particular interest in this study where information about asset attributes and prices for a large number of firms trading in the United States will be used to estimate the pricing functional. The existing literature, particularly the work by Ramezani (1991), provide a guide for selecting the appropriate attributes (quantitative and qualitative). These attributes – accounting, financial, and other information – are taken from the COMPUSTAT and CRSP financial data files. Table 1 lists the category of attributes that will be considered.

To obtain reasonable estimates of the functions in equation (4), the explanatory variables must be selected such that the resulting model is parsimonious. Two procedures are envisioned for selecting the explanatory variables, i.e., the attributes or factors. The first is the variable selection techniques based on Gibbs sampling described in George & McCulloch (1993) and Casella & George (1992). The main thrust of their proposed methodology is to use probabilistic techniques to select a "promising" subsets of all possible explanatory variables. Such subset has a higher *posterior probability* to be the 'appropriate' predictors.

The second variable selection methodology was proposed in the context of the GAM and investigated by Breiman & Friedman (1985). Their methodology uses a forward stepwise procedures to select the explanatory variables. In the first pass the bivariate relationship between the dependent variable and each potential explanatory variable is considered and the predictor that yields the highest R^2 is retained. The process is then repeated while retaining the explanatory variable selected in the first pass. This forward selection procedure is continued until the best predictor of the next pass increases the R^2 of the previous pass by less than some critical value, say 0.001.[4]

The AVAS algorithm applied to actual price and attribute data will yield estimates of the true underlying functions, $\hat{\theta}_i(q_i)$ and $\hat{F}_i(b_{ij})$'s, evaluated at each data point. The output is a vector of data for each function representing the 'optimal' transformations obtained by the AVAS algorithm. Using these transformations and the actual value of the dependent and the explanatory variables we obtain the residuals for the AVAS model, ϵ. Two procedures are envisioned for testing the additive non-parametric models proposed above, (4), against the standard parametric asset pricing models described in Appendix A.

The first involves classical goodness of the fit tests based on comparison of the residuals of the AVAS model and a well defined parametric model with the same explanatory and dependent variables (Judge, Griffiths, Hill & Lee 1980). The second procedure for testing the AVAS against well specified parametric models is to use the non-nested tests proposed by Delgado & Stengos (1994). These authors propose a specification test of a parametrically specified model against a non-nested alternative that is estimated using non-parametric regression techniques. The test statistic they develop uses the nearest neighbor regression as the (non-parametric) alternative specification. Their test statistic must be derived for the AVAS as the competing alternative model. Preliminary examination suggests that this will be a straight forward exercise.

The onset of the deadline to submit the AFE proposal has caused the presentation in this section to be heuristic and non-technical. The author has experimented with the estimation of AVAS with financial data (COMPUSTAT) and has worked out the mathematical details of the non-nested tests and has obtained most promising results. Unfortunately, however, detailed presentation of these ideas must be postponed to future version of this manuscript.

Summary and Extensions

In this proposal I developed a model of portfolio behavior which subsumes other asset pricing models as special case. The model suggests that because of nonlinearity in individual preferences and portfolio attribute relationships, as well as market imperfections such as taxes and transaction costs, asset pricing functionals will be highly nonlinear. A non-parametric procedure to uncover these nonlinearities was proposed. Cross-sectional data on stock prices and their attributes and time series data on option prices and their attributes have been used to conduct preliminary tests of the proposed model and the results have been generally very encouraging.

APPENDIX A:
The Attribute Model as a Unifying Framework

The general attribute model proposed above contains as its special case most utility based portfolio choice models. It therefore provides a way to assess the empirical validity of different models by using a nested test. To make this point clear we consider a number of existing models:

In the **state-preference** model of Arrow (1964) : Choose x to max $u(Z) = u(w_s)$ subject to $w_s = p_s x$: implies $p_i = \sum_{s=1}^{S} \theta_s p_{is}, \quad \forall\, i = 1, ..., N$ Here θ_s is the Arrow-Debreu price of the payoff in state s.

The **parameter preference** model (PPM) due to Markowitz (1952), Rubinstein (1973) and others : $u(Z) = u(m, v)$ where $m = \int_{\delta}^{\gamma} w\, dF(w)$, $v = \int_{\delta}^{\gamma} |w - \mu|^{\alpha} dF(w)$, and $F(.)$ is the wealth distribution. For the mean-variance model ($-\delta = \gamma = \infty$, $\mu = m$ and $\alpha = 2$) the optimization implies :

$$p_i = \left(\frac{1}{\theta}\frac{\partial u}{\partial m}\right) \frac{\partial m}{\partial x_i} + \left(\frac{1}{\theta}\frac{\partial u}{\partial v}\right) \frac{\partial v}{\partial x_i}$$

$$p_i = \theta_m \mu_i + \theta_v \sum_{j}^{N} \sigma_{ij} x_j$$

where μ_i is the expected price of the i-th asset, σ_{ij} is the covariance between the i-th and j-th asset prices, θ_m and θ_v are the shadow prices of the portfolio mean and variance, and θ is the marginal utility of wealth.

The **capital asset pricing model** (CAPM) of Sharpe (1964) is the market equilibrium version of PPM of Markowitz:

$$p_i = \theta_1 \mu_i + \theta_2 \beta_{iM}$$

where $\theta_1 = [1 + r_f]^{-1}$, $\theta_2 = -\theta_1 [\mu_M - (1 + r_f) p_M]$, $\beta_{iM} = \sigma_{iM}/\sigma_M^2$, r_f is the risk-free rate of interest, and p_M and μ_M are the current and the expected (end of period) values of the market portfolio.

The standard CAPM has been extended in a number of ways; The recognition that investors may be concerned with other variables in addition to the mean and variance has led to the development of the K-parameter versions of CAPM. A particularly interesting version of the K-factor model is due to Rubinstein (1973), who defines preferences over the n moments of wealth distribution. The first order necessary conditions, which have been aggregated over investors, include shadow prices with respect to the n moments of the wealth distribution and are analogous to the attribute model.

Others have considered factors other than those characterizing the returns distribution. Shanken (1985) considers liquidity, defined as the differential in the cost of buying and selling assets, or their bid-ask spread, as an important parameter effecting portfolio decisions. Denoting this factor by l_i, he derives the equilibrium condition for this version of CAPM as:

$$p_i \ = \ \theta_1\mu_1 \ + \ \theta_2\beta_{iM} \ + \ \theta_3 l_i \tag{4.5}$$

Equation (4.5) defines the security market 'plane' in an efficient market. Given β_{iM}, the greater the bid-ask spread, the lower the expected price, and given μ_1, the greater $\beta_i M$, the greater the liquidity. An example of other factors that influence preferences are taxes. Brennan (1971), Litzenberger & Ramaswamy (1970), Litzenberger & Ramaswamy (1982), and others have integrated tax considerations into the CAPM. The motivation for these models is the observation that, because of differential taxes, individuals may prefer capital gains to dividends.

Brennan proposed a version of the CAPM that accounts for the taxation of dividends with constant individual tax rates. Litzenberger & Ramaswamy (1982) extended this model to account for progressive taxation. These refinements bring other factors to bear on asset prices in the context of a single period portfolio behavior model – a very strong assumption that partly explains the empirical failure of these models.

The **The Inter-temporal Asset Pricing Models** improve upon the single-period CAPM by allowing multi-period portfolio decisions and re-balancing. Merton (1973) extended the simple CAPM to an inter-temporal setting in which the investment opportunities set evolves stochastically. Building on Merton's model, Breeden (1979) allowed the consumption opportunities as well as investment opportunities to be stochastic. Below we briefly demonstrate the consistency of the attribute framework with these inter-temporal models

In Merton's model the stochastic relation between the state variables is determined by a multidimensional Ito process. The state variables considered include the current level of wealth $w(t)$ and a vector of state variables, $S(t)$, which characterizes the changing investment opportunities. The vector $S(t)$ contains the current and expected asset prices, as well as their standard deviations.

Let $J(w(t), S(t), t)$ be the indirect utility function of wealth resulting from following an optimal consumption-investment strategy, $\forall \ t \ \in \ [t, T]$. Using Bellman's principle of optimality, Merton shows that at each point in time, $J(.)$ satisfies the following second-order partial differential system:

$$Max\ [u(c,t)\ +\ J_w\ m\ +\ J_t\ +\ \sum_{k}^{K-2} J_k\ n_k$$
$$+\ \frac{1}{2} J_{ww}\ v\ +\ \sum_{k}^{K-2} \sum_{i}^{N} J_{kw}\ \eta_{ik} x_i + \frac{1}{2} \sum_{k}^{K-2} \sum_{l}^{K-2} J_{kl}\ s_{kl}]\ =\ 0 \tag{4.6}$$

where

$$m\ =\ \sum_{i}^{N} (\mu_i\ -\ (1+r_f) p_i) x_i\ +\ (r_f w - c)$$

is the expected value of the portfolio,

$$v\ =\ \sum_{i}^{N} \sum_{j}^{N} \sigma_{ij} x_i x_j$$

is the portfolio variance, σ_{ij} is the covariance between the i^{th} and the j^{th} asset prices, n_k is the expected value of the k^{th} element of the state vector S(t), s_{kl} is the covariance between the k^{th} and l^{th} elements of S(t) and η_{ik} is the covariance between the i^{th} price and k^{th} element of S(t). The first-order conditions derived from (4.6) are:

$$u_c\ =\ J_w \tag{4.7}$$

$$J_w[\mu_i\ -\ (1+r_f)\ p_i]\ +\ J_{ww} \sum_{j}^{N} \sigma_{ij} x_j + \sum_{k}^{K-2} J_{kw} \eta_{ik}\ =\ 0\ \ \forall\ i \tag{4.8}$$

Equation (4.7) implies that the optimal consumption is determined by equating the marginal utility of current consumption and wealth (this is an inter-temporal envelope condition). Inverting (4.8) the asset demand functions are obtained;

$$x_i = -\ (J_w/J_{ww})\ \sum_{j}^{N} {\sigma_{ij}}^{-1} (\mu_i\ -\ (1+r_f)\ p_i)$$
$$-\ \sum_{k}^{K-2} (J_{kw}/J_{ww})\ \sum_{j}^{N} {\sigma_{ij}}^{-1}\ \eta_{jk} \tag{4.9}$$

Merton's model shows that in an inter-temporal setting there will be two components to the demand for assets; First the conventional demand, as in the single-period mean-variance model, and second a hedge against the adverse effects of the state variables, which act through their covariance with prices.

Note that (4.9) can be solved for p_i as a function of variance-covariance terms, r_f and other variables to obtain the relation between asset prices and the attributes:

$$p_i = \theta_1 \mu_i \; + \; \sum_j^N \theta_j \sigma_{ij} + \sum_k^{K-2} \theta_k \eta_{jk} \tag{4.9'}$$

where $\theta_1 = [1 + r_f]^{-1}$, $\theta_j = [\theta_1 J_{ww} J_w^{-1}] x_j$, and $\theta_k = [\theta_1 J_{kw} J_w^{-1}]$. Again it is simple to determine the attributes which would give rise to a pricing relationship similar to the inter-temporal CAPM.

Similar results can be established using Breeden's model, in which consumption opportunities are also stochastic.[5] Breeden points out that in practice it may be difficult to identify the relevant (K-2) state variables. He shows that the multi-beta model is equivalent to a single-beta model in which aggregate consumption is the only state variable. He argues that correlation between asset prices and aggregate consumption is a more appropriate measure of risk than the correlation between asset prices and aggregate wealth.

When consumption opportunities are stochastic, consumption has the form $c = c(w(t), S(t), t)$. From the first order conditions above we have $J_{ww} = u_{cc}\, c_w$ and $J_{wk} = u_{cc}\, c_k$. Substituting these into (4.8) and rearranging we obtain:

$$T_c[\mu_i \; - \; (1 \; + \; r_f) \; P_i] = \; \sigma_{iw} \; c_w \; - \; \sum_k^K \; \eta_{ik} \; c_k \tag{4.10}$$

where $T_c \;=\; -u_c/u_{cc}$ is the individual's absolute risk tolerance defined on consumption. From c(w,x,t) we also have; $dc \;=\; c_w dw \;+\; \sum_k^K \; c_k dS_k$, which shows that changes in consumption are linearly related to changes in wealth and the state variables. Multiplying this expression by p_i and taking expectations gives:

$$\sigma_{ic} \;\;=\;\; \sigma_{iw} c_w \;+\; \sum_k^K \;\; \eta_{ik} c_k \tag{4.11}$$

This allows us to substitute for σ_{ic} in (4.9). With this substitution we see that optimal portfolio choice requires that the covariance of each asset price with optimal consumption is proportional to that asset's expected excess return. The price relation obtained from the counter part of (4.9) for the the inter-temporal Consumption CAPM is:

$$p_i \ = \ \theta_1 \mu_i \ + \ \theta_2 \beta_{ic} \tag{4.12}$$

where

$$\theta_1 \ = \ [1 + r_f]^{-1},$$

$$\theta_2 \ = \ -\theta_1 \ [\mu_M \ - \ (1 + r_f) p_M)] / \beta_{Mc},$$

$\beta_{ic} \ = \ \sigma_{ic}/\sigma_c^2$, and $\beta_{Mc} \ = \ \sigma_{Mc}/\sigma_c^2$ are the asset and consumption betas.

Breeden argues that in equilibrium, the risk associated with an asset may be represented by a single aggregate consumption beta. This is an important simplification relative to the multi-beta relation.[6] The equilibrium pricing relation in (4.12) is clearly an attribute pricing model, in which two principal characteristics, μ_i and β_{ic} determine the returns on asset i.

The **Accounting Valuation Models** originating in the accounting literature associate asset prices (firm value) with the information contained in financial statements. Accounting models, similar to arbitrage pricing models, are not based on models of investor preferences. Asset prices are assumed to depend upon discounted future earnings of the asset. It follows from this causal relation that asset prices are related to factors which influence expected earnings.

Based on this reasoning, most accounting models simply assume that asset prices are functions of information variables. A variety of models based on this premise have appeared in this literature. Some important work includes Miller & Modigliani (1961), Beaver, Lambert & Morse (1980), Ohlson (1988), and Ohlson (1989).

In the highly celebrated 'clean surplus' model of Ohlson (1988), the market value of firms' common stocks at any point in time, p_t is assumed to be a linear function of earnings realized during the past period e_t, the book value y_t, dividends per share d_t, and a vector of 'other' value relevant information, v_t:

$$p_t \ = \ \theta_1 e_t \ + \ \theta_2 y_t \ + \ \theta_3 d_t \ + \ \theta_4 v_t \tag{4.13}$$

The Miller-Modigliani dividend irrelevancy theorem (Miller & Modigliani 1961) states that changes in the book value of a firm are off set by its dividend payments. Since asset prices will be reflective of book values, dividend policy should not effect prices. The *Clean Surplus Equation*, $y_t \ = \ y_{t-1} \ + \ e_t \ - \ d_t$ is a consequence of this theorem and may be substituted in (4.13). This substitution permits one to eliminate dividends and focus solely on accounting earnings, book value, and other variables as determinants of prices.

Future values of these variable are assumed to be generated by a 'linear information dynamics' (a Markovian stochastic process). This enables the researcher to obtain an estimate of the expected value of explanatory variables. Assuming risk neutral agents, the expected price of the asset will be determined by the expected values of these variables and the θ's. Amir (1991) provides an empirical examination of this model.

The attribute model provides an important justification for relating asset prices to value-relevant signals. However, the model also suggests that the relation between prices and the value-relevant variables will *not* necessarily be linear (see Das & Lev (1990)).

APPENDIX B: The Details of the AVAS Algorithm

The AVAS specification assumes that an arbitrary function of the dependent variable, $\Theta(N)$, is related to functions of the independent variables, $F_i(S_i)$, via an additive structure of the form:

$$\Theta(N) = \sum_{i=1}^{n} F_i(S_i) + \epsilon \qquad (A1)$$

where the subscript i refers to the i-th explanatory variable, and it is assumed that $\Theta(N)$ is monotone and strictly increasing, $F_i(S_i)$'s have a multivariate normal distribution, ϵ is normally distributed with mean zero, and ϵ is independent of S_i's. Both the dependent and the explanatory variables may be categorical or continuous (Tibshirani 1988).

The AVAS algorithm utilizes general statistical procedures to obtain 'optimal' estimates, $\hat{\Theta}(N)$ and $\hat{F}_i(S_i)$, for $\Theta(.)$ and $F_i(.)$. The estimated functions, $\hat{\Theta}(.)$ and $\hat{F}_i(.)$, are 'optimal' in the sense that they minimize the expected squared residuals:

$$E\ [\ \Theta(N)\ -\ \sum_{i}^{n}\ F_i(S_i)\ ]^2 \qquad (A2)$$

subject to $\Theta(N)$ having a constant conditional variance:

$$VAR\ (\ \Theta(N) \mid \sum_{i}^{n}\ F_i(S_i)\ )\ = Constant,$$

and $E\ \Theta(N) = 0, VAR\ (\ \Theta(N)\ ) = 1$. The restriction on the conditional variance achieves constant residual variance. $\Theta(N)$ and $F_i(.)$'s are standardized to remove the effects of measurement scale and to ease interpretation.

The algorithm for implementing AVAS works in an alternating manner. In the first step, a functional form for $\Theta(N)$ is assumed, usually $\hat{\Theta}(N) = \frac{(N-E(N))}{\sigma_N}$, where E denotes expectation and σ_N is the standard deviation of N. Conditional on this form for $\Theta(N)$, a search for functions $\hat{F}_i(S_i)$ that minimize the expected squared residuals and lead to constant conditional variance is conducted. This step leads to estimates $\hat{F}_i(.)$'s, conditional on which, the algorithm searches for new specifications for $\hat{\Theta}(.)$ that would further minimize the expected squared residuals while meeting other restrictions in (A2). This alternating process – known as backfitting – continues until convergence is achieved and the expected squared residuals are minimized. The Proofs related to existence of these functions and convergence of the algorithm are beyond the scope of this appendix but can be found in Hastie & Tibshirani (1990).

It is worth noting that the class of functions estimated can be quite large, including linear forms, various quadratic forms, and a wide range of other specifications. At each step of the algorithm, any of the available non-parametric curve estimation methods including the running mean, locally-weighted running-line, kernel and cubic splines – collectively known as smoothers – may be used to obtain the estimates $\hat{\Theta}(.)$ and $\hat{F}_i(.)$'s. Once the algorithm has converged, plots of the optimal functions generated by the "smoothing" procedures provide graphical representation of the relation between the dependent and individual explanatory variables.

The preceding discussion of GAM and AVAS is, of course, incomplete. In particular, no attempts were made to discuss statistical inference and forecasting within this framework. A thorough account of these and other developments in this area is beyond the scope of this paper. Hastie & Tibshirani (1990) present a comprehensive discussion of these issues. They argue, that AVAS is a useful method for model building and general exploratory analysis that should precede standard regression analysis. The AVAS and other additive models are now widely available as part of various statistical packages and can be used for data analysis with little programming effort.

Notes

[1] Cochran & Hansen (1992) page 10.

[2] Assumptions on $u(\, . \,)$ and $G(\, . \,)$ insure that the second order sufficient conditions for a maximum are met and the constraints are qualified.

[3] Recently popularized Kernel estimators are developed in great detail by Hardle (1989) and Silverman (1986) and are nicely surveyed by Altman (1992).

[4] Baron-Adesi and Talwar 1983 show that asset pricing equations with a larger number of explanatory variables, are more likely to be homoscedastic. This suggests that the distributional properties of the residuals of the selected model should be studied using tests for heteroscadasticity and normality.

[5] The discussion here generalizes to Breeden's (1984) many consumption goods model as well.

[6] Cornell (1979) criticized Breeden's model. I should cite this here though his criticism has no relevance for our purposes.

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